EFFECT OF GEOMETRY ON THERMAL STABILITY

By

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

2004

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To my beloved family.

ACKNOWLEDGMENTS

I wish to express my deep gratitude to my advisors, Dr. Lewis E. Johns and Dr. Ranga Narayanan, for their enormous help, encouragements and guidance throughout this work. I am thankful to the other members of my supervisory committee, Dr. Jason Weaver and Dr. David Mikolaitis for their time and effort; and to the Chemical Engineering Department at the University of Florida for the excellent education they provided throughout my graduate studies. I am grateful to my family in Bahrain for their encouragement, patience and prayers. This work was made possible by the heartfelt love and support of my wife, and the peaceful serenity of our daughter, Najlaa. I would also like to thank my friend Qais for being a true brother who made me feel at home while studying in the United States. I would also like to thank my friends in Bahrain for always being in touch over the years of my study. Finally I gratefully acknowledge the University of Bahrain and the people of Bahrain, whose financial support made it possible for me to carry out my graduate studies in the United States of America.

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LIST OF SYMBOLS

Symbol	Definition	Dimension
C_v	Constant volume specific heat capacity of the material in space	$L^2.t^{-2}.T^{-1}$
C_v^C	Constant volume specific heat capacity of the central layer or the rod dimension	$L^2.t^{-2}.T^{-1}$
$\vec{g}$	Gravitational acceleration	$L.t^{-2}$
k	Thermal conductivity of the material in space	$M.L.T^{-1}.t^{-3}$
k_c	Thermal conductivity of fluid filling the pores (Chapter 6)	$M.L.T^{-1}.t^{-3}$
k^C	Thermal conductivity of the central layer	$M.L.T^{-1}.t^{-3}$
k_d	Thermal diffusivity	$L^2.t^{-1}$
k_p	Permeability of porous medium	L^2
κ	Constant ≥ 1	-
L	Distance between the plane wall and the geometry center	L
$\vec{q}$	Heat flux	$M.t^{-3}$
Q	Rate of heat generation per unit volume	$M.L^{-1}.t^{-3}$
Q_f	Total volumetric flow rate	$L^3.t^{-1}$
Ra	Rayleigh number	-
T	Temperature rise in the material in space	T
T^*	Scaled temperature rise in the material in space	-
$\bar{T}$	Characteristic temperature	T
T^C	Temperature rise in the central layer	T

T_C	Low temperature	T
T_H	High temperature	T
$T_{\max}$	Maximum temperature	T
T_{steady}	Steady state temperature in the material in space	-
T_{steady}^{C}	Steady state temperature in the central layer or the rod	-
T_w	Temperature of the plane wall	T
t	Time	t
t^*	Scaled time	-
$\bar{t}$	Characteristic time	t
x	Dimension at which plane walls and central layer move	L
α	Coefficient of thermal expansion (in Chapter 6)	T^{-1}
β	One half of the central layer thickness (in Chapter 3)	L
ε	Distance by which the central layer or the rod is shifted	L
θ	Geometric angle	[ ]°
μ	Viscosity of the material in space	$M.L^{-1}.t^{-1}$
μ_w	Viscosity of the material at $T = T_w$	$M.L^{-1}.t^{-1}$
ν	Kinematic viscosity	$L^2.t^{-1}$
ρ	Density of the material in space	$M.L^{-3}$
ρ^C	Density of the material of the central layer or the rod	$M.L^{-3}$

Abstract of Dissertation Presented to the Graduate School
of the University of Florida in Partial Fulfillment of the
Requirements for the Degree of Doctor of Philosophy

EFFECT OF GEOMETRY ON THERMAL STABILITY

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December 2004

Chair: Lewis E. Johns
Major Department: Chemical Engineering

Our study focused on the effect of geometry change on the thermal stability of heat-producing regions wherein the heat source depends linearly on temperature. Thermal stability was studied in selected rectangular and cylindrical geometries. We defined thermal instability or thermal explosion by an infinite maximum temperature, which in turn translates into our inability to find a steady solution.

To understand the effect of geometry change on the thermal stability of the examples considered, calculations are provided using the techniques of domain perturbation. In solving several introductory problems, we found that thin heat-producing regions are thermally more stable than thick heat-producing regions. By considering thermal stability in an elliptical region we found that such regions are thermally more stable than companion circular regions of the same area and thereby showing that the thermal stability depends strongly on the geometry.

We studied the effect of boundary-condition change on the thermal stability where heat is transferred only by conduction. We found that for two-point boundary value

problem the system is always more thermally stable for isothermal-isothermal boundary conditions than for isothermal-adiabatic and isothermal-perfectly conducting boundary conditions. We also studied the effect of change in the heat-transfer mode as a little forced convection is added to a problem of a heat-generating annular region at constant-wall temperature. We found that due to convection this problem always becomes stable thus the thermal stability of this problem depends strongly on the heat-transfer mode.

As the theme of our study is on geometric effect on thermal stability we considered the effect of geometry change on fluid-flow problems in which the heat source arises from viscous dissipation. Here too we found that the heat-transfer rate depends on the geometry implying that the thermal stability must depend on geometry as well.

We also studied the effect of geometry change in a porous media with heat generation and natural convection. We found that an adverse temperature gradient can induce natural convection and that the occurrence of convection depends strongly on the shape of the geometry.

CHAPTER 1
INTRODUCTION

Our study focused on the effect of the shapes of heat-producing regions on their thermal stability. Heat-producing regions wherein the heat generation is a positive function of the temperature are encountered in a variety of chemical engineering applications. One example is in the operation of chemical reactors with exothermic reactions. The thermal stability of these regions becomes an issue when the rate of heat production exceeds the rate of its removal (a condition known as thermal runaway[1] or ignition).

Now it has been well established that thermal runaway of different materials is dependent on the shape of the region, its size, the ambient temperature, and the physical properties of the material and the surroundings. In other words, for any given geometry, there is a critical size above which the heat dissipated to the surroundings cannot keep up with the heat generated inside the material.

1.1 Literature Review

Thermal runaway phenomena have been studied extensively, most notably by Semenov (1959), Frank-Kamenetskii (1969), Zeldovich *et al.* (1985) and Gray and Scott (1994) in connection with thermal explosions; and by Bilous and Amundson (1955-1956), Barkelew (1959), Perlmutter (1972) and Fogler and Crynes (1985) in relation to reactor stability. Several studies were done by Biesenberger and Sebastian (1975-1983) in the field of polymerization reactions; and by numerous others in the area of combustion and flames (Biesenberger and Sebastian 1983).

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[1] It is also known as thermal explosion.

To show why this problem continues to be of interest and what sorts of problems we intended to investigate, we first discuss a simple problem to show how the geometry change can affect the thermal stability. Then we introduce the problems of interest and their physics.

Let us consider an annular region of a heat-producing substance where the inner wall and the outer wall are held at fixed temperatures (Figure 1-1A). A typical undergraduate student would know how to calculate the heat transported, in the case where there is no heat production in the annular region, and the inner and the outer walls are held at different temperatures. Now if the inner cylinder is displaced by an amount ε as shown in Figure 1-1B, it is clear at once that the heat-transfer rate can change only at order ε^2. It is less obvious, but no less true, that the heat-transfer rate increases. Chapter 2 gives the detailed solution for such a case.

Now return to the heat generation problem, and imagine that the annular region produces heat, such that the rate of heat production is a positive function of the local temperature; and further imagine that the walls are held at the same fixed temperature. The heat released in the material may be produced in a variety of ways[2], but must be an increasing function of the temperature of the material. To keep this system stable, the heat generated must be removed from the system at the rate at which it is produced. Assume that this heat is removed by conduction to the outer wall, which serve as a heat sink. To conduct the heat away, the temperature of the material must rise above

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[2] The heat can be produced via exothermic reaction, viscous heating, laser pulse, electron beam, etc.

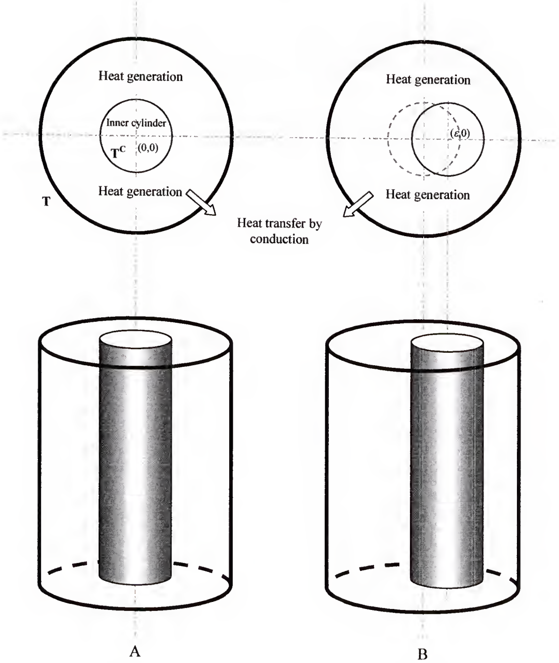

Figure 1-1. Annulus with heat-producing material. A) On-center. B) Off-center.

that of the sink. The more distant a point of the material is from the boundary, the higher its temperature must rise; but the higher the rise, the stronger the source, creating positive feedback that might rule out a steady solution.

One issue is whether the heat produced can leave quickly enough, or whether thermal instability might set in. In particular it becomes questionable whether the thermal stability is advanced or retarded for the case of a distorted annulus, when compared to a symmetric annular region. We might initially ask two questions: the first is of research interest and the second is of technological interest. First, why should the delay of the thermal stability be in doubt? Second, for what sort of problems would this geometry be of interest? To answer the first question, accept for now the fact that a shift in the annulus causes the heat-transfer rate by conduction to increase as azimuthal gradients can now come into action.[3] But also remember that the shifted inner cylinder creates both a large and a small heat-producing region. The large region determines the fate of the thermal stability; and if the azimuthal temperature gradients from this region cannot meet the demand that heat must be removed fast enough, then thermal instability will set in, and this instability will be manifested by a temperature surge. Only a calculation can tell whether the rate of heat production in the distorted annular compartment will be higher than the enhanced heat-transfer rate, or the other way around. This calculation is presented in Chapter 4. To motivate the calculation, a simple problem is taken up initially. Through the working of this simple problem, we can learn which methods and techniques are needed to be able to solve the proposed, more complicated problem. This simple problem is given in Chapter 3. Now let us answer the second question by demonstrating some applications to this research.

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[3] Chapter 2 demonstrates this fact for a model problem.

1.2 Applications of Thermal Explosion and Thermal Stability Phenomena

There are many applications where thermal explosion analysis and thermal stability studies are important, and where annular geometries are encountered. One of the common applications is polymer manufacturing.[4,5] An interesting phenomenon known as *hydrodynamic thermal explosion* occurs. This phenomenon is important in the flows of polymer melts. It can be illustrated by considering an established flow of polymer melt in an infinite tube. Assume that the thermal resistance of the tube walls is low, so that the walls are maintained at a constant temperature equal to that of the surrounding medium. For manufacturing purposes the polymer must be at a certain temperature range in which it can flow without losing its main structural properties (Figure 1-2A). If, at certain conditions, the viscous heat[6] arising from the polymer flow increases at a higher rate than that dissipated to the surroundings through the tube walls, then the temperature begins to rise, and the material starts to soften and decompose. Under such conditions, the desired flows of polymer melts become impossible in a pipe (Zeldovich *et al.* 1985). This situation is fully analogous to thermal explosion. To dissipate the heat fast enough to the surroundings, the polymer flow reactor might typically be in the form of a shell and tube heat exchanger (Figure 1-2B). The geometry of the equipment undoubtedly affects the thermal stability of the process.

---

[4] Polymerization reactions are typically strongly exothermic. Moreover, polymers are typically poor thermal conductors.

[5] Polymer manufacturing is one of the fastest growing industries worldwide. According to a recent study, it is expected that US demand of six leading thermoplastic polymers alone (polypropylene, polyethylene, PVC, polystyrene, LDPE and HDPE) will exceed 15 billion pounds by the year 2006 (Weizer,W., 2002, Plastic Film to 2006, Freedonia Industrial Studies, Study # 1588, The Freedonia Group, Inc., Ohio.). Global demand is already higher than 100 billion pounds (Tirrel, M., Galvan, R. and Laurence, R.L., 1987, Polymerization Reactors, In: Chemical Reaction and Reactor Engineering, Editor: Carberry, J.J. and Varma, A., Marcel Dekker, Inc., New York.).

[6] The frictional heat due to shear between adjacent fluid layers.

Typically, the interaction between viscous heating and flow is of importance in a number of applications involving flow of materials with temperature-dependent properties. These include lubrication, food processing, and polymer processing (Caridis *et al.* 1997).

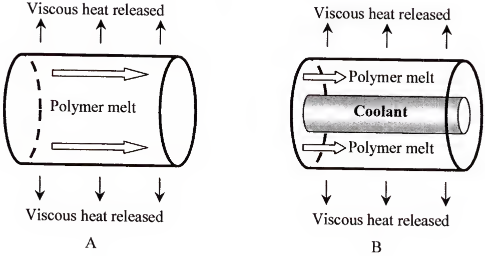

Figure 1-2. Polymer manufacturing example. A) Without cooling rod. B) With cooling rod.

In addition to the above applications, the thermal runaway phenomenon is one of the main issues considered in the design of industrial equipment (*e.g.*, in the design of heat exchangers and reactors). To build safe industrial plants, designers must consider all the factors that might lead to thermal explosions, (the shape of the region, its size, the ambient temperature, the physical properties of the material and the surroundings, etc.).

Aside from the earlier examples, the thermal runaway phenomenon plays an important role in dam construction. We know that dams require gigantic blocks of concrete that are linked together. We also know that concrete reacts exothermically with water for long periods of time. Thus, based on the volume of the concrete used in dam construction, it could take tens of decades for the concrete to cool and properly cure without engineering intervention. To overcome this problem and to maintain the thermal stability of the dam, dam builders dissipate the heat generated by the concrete by imbedding an extremely long one-inch steel pipe through the interconnecting concrete blocks to circulate cold water.

The thermal stability features of the problems discussed above are known for several regular geometries (*e.g.*, planar and cylindrical geometries), and for several kinds of heat generation. In our study, the case in which heat-generation function depends *linearly* on temperature shows interesting results for a simple problem; then, more practical problems and conditions are investigated. A case in which heat-generation function depends linearly on temperature is a good model for problems with viscous fluid flows. In these problems, the fluidity (which is the reciprocal of the viscosity) depends linearly on the temperature of the fluid over a fair temperature range. The effect of geometry change on thermal stability for such problems is discussed in Chapter 5.

Before going to the next chapter, let us give an idea about the consequences of thermal runaway and show some of its effects. Thermal runaway can cause huge industrial, economic, and environmental damage. The physical consequences of thermal runaway, which are the reasons behind many industrial and environmental disasters, can be summarized as follows (Gustin 1991):

- An increase of the rate of the chemical reactions.
- The occurrence of unwanted exothermic reactions that are not obtained in normal process conditions. Usually these reactions are decomposition reactions of the reaction mixture.
- A pressure increase that often causes the reactor vessel to burst due to two phenomena:

  - An increase of the reaction-mixture vapor pressure due to the temperature rise.
  - The production of noncondensable decomposition gases.

Now let us give a catastrophic example caused by thermal runaway reactions. This example (the worst disaster in Texas history) occurred in 1947, when a French ship exploded at the docks in Texas City. The storage tanks of the French vessel were filled with ammonium nitrate, a fertilizer. For some reason, the temperature of ammonium nitrate began to rise, and then it started to react and produce heat. Because of the large dimensions of the storage tanks, the rate of heat removal could not meet the rate at which it was being produced. Because of this, a thermal runaway reaction took place, and an explosion occurred. This thermal explosion led to about 600 tragic fatalities, and destroyed the entire dock area, along with the nearby buildings and facilities. From this example, we can see how the size of a heat-producing region can affect its thermal stability.

There are many more examples of effect of the geometry of the heat-producing region becoming a possible reason for thermal explosion to occur. One example is a chemical incident that occurred in Seveso, Italy, in 1976, at a Union Carbide factory. In this accident, the thermal runaway led to an immediate contamination of some 10 square miles of land and vegetation. More than 200,000 people had to be evacuated from their homes and were treated for dioxin poisoning, and as many as 42000 people needed medical follow-up for more than 20 years (Caridis *et al.* 1997). Another example is the famous incident that occurred at a Union Carbide chemical plant in Bhopal, India in 1984. This disaster was particularly notable as a thermal runaway process, as it led to more than 3000 tragic fatalities (Mcintosh *et al.* 2001). The last example is another chemical accident at a Sandoz warehouse in Basel, Switzerland, in 1986. In this accident, thermal runaway caused massive pollution of the Rhine, and the death of more than a half million fish (Caridis *et al.* 1997).

From these real examples, one can see how dangerous the effects of thermal runaway could be, and also why it is still important to study this phenomenon.

CHAPTER 2
HEAT TRANSFER IN AN ANNULUS WITH NO HEAT PRODUCTION

In this chapter we show how to solve an auxiliary problem (*i.e.*, heat transfer in a distorted annulus without heat generation). This simple problem is used to motivate us, and prove that thermal stability could be affected by changes in geometry. In this problem, the total heat-transfer rate, Q, is calculated in an annular region when there is no heat production, and the inner and the outer walls are held at different temperatures (Figure 2-1). The total heat-transfer rate is calculated for a perfectly centered annulus, and for a distorted annulus. Results from each of the cases are compared and a conclusion is drawn.

Before the calculations for this simple problem are presented and discussed, let us examine what might be going on from a picture argument. Such an argument might help in predicting conclusions and verifying calculations. Figure 2-1A shows two concentric cylinders at different surface temperatures. The surface temperature of the inner cylinder, T_H, is greater than that of the outer cylinder, T_C. Let us assume that the heat is transferred only by conduction through the annular region, and further assume that no heat is generated in the annular region. Based on how the problem is defined, it is obvious that no heat can accumulate in the annular region. Therefore, the total heat transferred to the surroundings must be equal to the total heat released into the annular region by the hot inner cylinder.

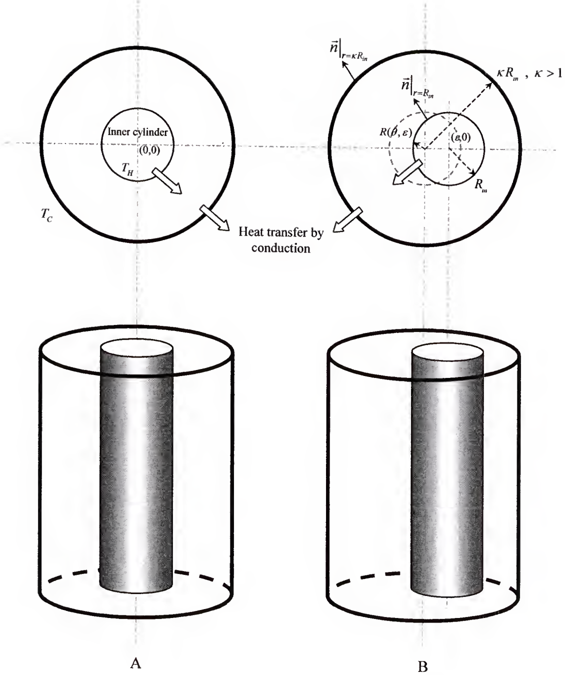

Figure 2-1. Annulus without heat-producing material. A) Centered. B) Distorted.

Begin with the case of a perfectly centered annulus (Figure 2-1A). The heat released into the annular region will be conducted equally in the radial direction via the radial temperature gradient (*i.e.*, the temperature gradient in the r direction), see Figure 2-2A. For the distorted annulus (Figure 2-1B), the shifting of the inner cylinder creates both a thick and a thin region. Therefore, the heat released into the annular region cannot be conducted equally in the radial direction, as in this case the radial temperature gradient is not evenly distributed around the inner cylinder. In this case the radial temperature gradient for the thin region will be higher than it is for the thick region. This is because, by shifting the inner cylinder to the right (Figure 2-1B), the distance between the rod wall and the pipe wall to the right of the rod will become shorter; and since the temperatures at the two walls are held fixed, the temperature gradient increases.

By the same measure, we can show that for the thick region, the temperature gradient decreases. As a result of this, the heat-transfer rate by conduction across the thin region ought to be higher than that for the thick region. And as we know, to conduct away the heat released into the annular region, the temperature at any point in this region must rise above that of the outer wall. The further a point in the region is from the outer wall, the higher its temperature must rise. From this we understand that, at any point at a fixed distance from the center of the inner wall, $\hat{R}$, (where $R_{in} < \hat{R} < \kappa R_{in}$), the temperature in the thick region will be higher than that in the thin region (Figure 2-3). From the second law of thermodynamics, we know that heat flows from high to low temperature. Based on this fact, the heat in the higher-temperature region (*i.e.*, the thick region), will start to conduct in the azimuthal direction toward the lower-temperature region (*i.e.*, the thin region), see Figure 2-2B. Thus, the heat-transfer rate by conduction will increase as the azimuthal gradients come into action. From this, we can conclude that the total heat-transfer rate for a distorted annulus is greater than that for a perfectly centered one. To verify this conclusion, the total heat-transfer rate is calculated for each of the two cases (shown next).

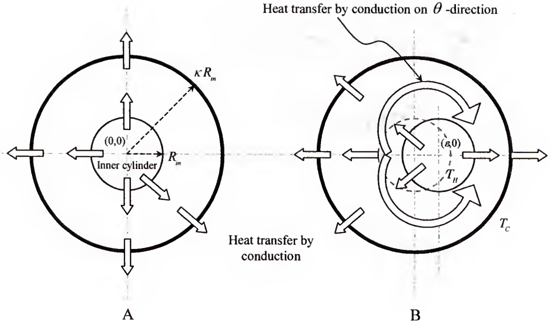
Heat transfer by conduction on θ -direction
κR_in
(0,0)
R_in
Inner cylinder
(ε,0)
T_H
T_C
Heat transfer by conduction
A
B

Figure 2-2. Heat dissipation in a perfectly centered annulus and in a distorted annulus (*no heat generation in the annular region*). A) Centered. B) Distorted.

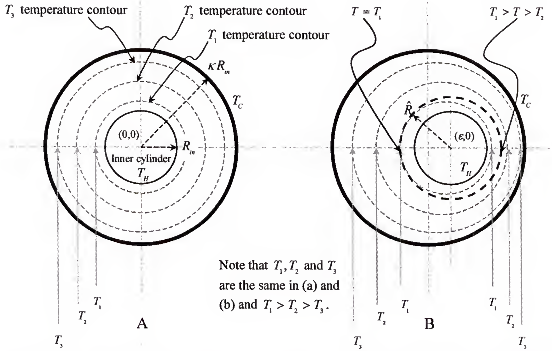

Figure 2-3. Temperature distribution in a perfectly centered annulus and in a distorted annulus (*no heat generation in the annular region*). A) Centered. B) Distorted.

Calculating the total heat-transfer rate. This section demonstrates the calculation of the total heat-transfer rate Q for the case of a perfectly centered annulus, and for the case of a distorted annulus. Then we compare the total heat-transfer rate Q obtained from each of the two cases, to show the difference.

Let us start with the left part of Figure 2-1. For this geometry, the total heat-transfer rate Q per unit length is obtained as

$$Q = \int dA\vec{n} \cdot \vec{q} \tag{2.1}$$

where $\vec{q}$ is the heat flux and $\vec{n}$ is the unit outward normal to the surface of the inner or the outer cylinder. Hence, because of radial symmetry, it is, (at the inner wall)

$$Q = -2\pi k R_{in} \left.\frac{dT}{dr}\right|_{r=R_{in}} \tag{2.2}$$

where R_{in} is the radius of the inner cylinder, and T is the temperature field.

To use Equation (2.2) the temperature in the annular region must be found. It is the solution to

$$0 = k\nabla^2 T = k\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)\right] \tag{2.3}$$

satisfying the boundary conditions

$$T\left(r = R_{in}\right) = T_H \tag{2.4}$$

and

$$T\left(r = \kappa R_{in}\right) = T_C \tag{2.5}$$

where $\kappa > 1$. Solving this problem, the temperature in the annular region is obtained as

$$T = \frac{\left(T_H - T_C\right)}{\ln\left(\kappa\right)}\ln\left(\frac{R_{in}}{r}\right) + T_H \tag{2.6}$$

and the rate of heat transfer at the outer wall or the inner wall of the symmetric geometry is obtained as

$$Q = 2\pi k \frac{(T_H - T_C)}{\ln(\kappa)} > 0 \tag{2.7}$$

Moreover,

$$\frac{dT}{dr} = -\frac{(T_H - T_C)}{\ln(\kappa)} \frac{1}{r} \tag{2.8}$$

$$\frac{d^2T}{dr^2} = \frac{(T_H - T_C)}{\ln(\kappa)} \frac{1}{r^2} \tag{2.9}$$

and

$$\frac{d^3T}{dr^3} = -2\frac{(T_H - T_C)}{\ln(\kappa)} \frac{1}{r^3} \tag{2.10}$$

These derivatives will be useful later on.

Now let us obtain the heat-transfer rate at *the inner wall* for the perturbed case, where the inner cylinder is displaced from the center of annulus by a distance ε (Figure 2-1B). In this case, the outer wall presents no problem; it is the inner wall whose surface is displaced. Let this surface be denoted as

$$r = R(\theta, \varepsilon) \tag{2.11}$$

Then, if X and Y denote the Cartesian coordinates of the points (R,θ) on the surface of the inner cylinder, there obtains

$$[X - \varepsilon]^2 + Y^2 = R_{in}^2 \tag{2.12}$$

where X and Y are given in terms of R and θ by

$$X = R\cos\theta \tag{2.13}$$

and

$$Y = R\sin\theta \tag{2.14}$$

whereupon R, as a function of θ and ε, must satisfy

$$R^2 - 2\varepsilon R\cos\theta = R_{in}^2 - \varepsilon^2 \tag{2.15}$$

Then, to obtain the total heat-transfer rate Q for this case, we need to know how to deal with this irregular[1] geometry (Figure 2-1B). Because this geometry is not symmetric, this leads us to the *domain perturbation* method[2,3] which can give a good approximation to the solution. To use this method, we must assume that the inner cylinder is shifted slightly from the center of the annulus (*i.e.*, ε is small). Then, the basic strategy is to obtain a solution valid at $\varepsilon = 0$ and then to add corrections terms for the small displacement. Appendix A gives detailed explanation for this method.

Therefore, the distorted annulus problem will be solved for a reference case that is the case where $\varepsilon = 0$, which is the symmetric case (Figure 2-1A). The distorted annulus problem is then solved by perturbing the dependent variables in this problem about the reference state.

Note that the reference case is the symmetric case that we solved earlier and the total heat-transfer rate Q for this case is given by Equation (2.7), namely by

$$Q = 2\pi k\frac{(T_H - T_C)}{\ln\kappa} > 0$$

---

[1] By the word irregular, we do not mean to imply not smooth; but instead, not convenient.

[2] A method used in calculations in both classical physics and quantum mechanics, in which the system is divided into a part that is exactly calculable and a small term, which prevents the whole system from being exactly calculable. The technique of perturbation theory enables the effects of the small terms to be calculated by infinite series (which is in general an asymptotic series).

[3] For further information about this method see Morse, P.M. and Feshbach, H., 1953, Methods of Theoretical Physics, McGraw-Hill, New York, or essay 2 in Johns, L.E. and Narayanan, R., 2002, Interfacial Instability, Springer, New York.

Then, to perturb the solution of this problem about the reference state we must expand R and T, in powers of ε and obtain problems in $Q_o, Q_1, Q_2, \ldots$ on the reference domain. Therefore, the expansion[4] of R, which denote the surface of the inner wall, can be written in the form

$$R = R_o + \varepsilon R_1 + \frac{1}{2}\varepsilon^2 R_2 + \ldots \qquad (2.16)$$

where

$$R_o = R_{in} \qquad (2.17)$$

and where R_1 and R_2 can be found by substituting Equation (2.16) into Equation (2.15), and expanding R^2 to the second order in ε, which gives

$$\left(R_o^2 + 2\varepsilon R_o R_1 + \varepsilon^2 R_1^2 + \varepsilon^2 R_o R_2 + \ldots\right) - \left(2\varepsilon R_o + 2\varepsilon^2 R_1 + \ldots\right)\cos\theta_o = R_{in}^2 - \varepsilon^2 \qquad (2.18)$$

From this we obtain at order ε^1

$$R_1 = \frac{dR}{d\varepsilon} = \cos\theta_o \ , \quad \text{at } \varepsilon = 0 \qquad (2.19)$$

and at order ε^2

$$R_2 = \frac{d^2R}{d\varepsilon^2} = -\frac{\sin^2\theta_o}{R_o} \ , \quad \text{at } \varepsilon = 0 \qquad (2.20)$$

while the surface of the outer wall does not change, which means

$$r = \kappa R_{in} \qquad (2.21)$$

and the temperature expansion in the domain, by using the rule given in Appendix A, will be

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[4] The expansions will be taken to the second order of ε *i.e.*, ε^2.

$$T(R,\theta) = T_o(R_o,\theta_o) + \varepsilon T_1(R_o,\theta_o) + \frac{1}{2}\varepsilon^2 T_2(R_o,\theta_o) + \dots \tag{2.22}$$

It is important to mention that the solution of the reference case will be denoted by affixing a subscript zero to its variables while the subscripts one and two will denote the expansions to first and second order of ε, respectively. Thus, the temperature in the annular region and the total heat-transfer rate for the symmetric case given by Equations (2.6) and (2.7), respectively, can now be written as

$$T_o = \frac{(T_H - T_C)}{\ln(\kappa)} \ln\left(\frac{R_{in}}{r_o}\right) + T_H \tag{2.23}$$

and

$$Q_o = 2\pi k \frac{(T_H - T_C)}{\ln(\kappa)} > 0 \tag{2.24}$$

Now, the total heat-transfer rate Q for the perturbed case is again determined via Equation (2.1), which is

$$Q = \int dA \vec{n} \cdot \vec{q}$$

Remember that for this case the calculation to obtain the total heat-transfer rate will be made at the inner wall.[5] As a result of this, the heat flux, $\vec{q}$, and the unit outward normal, $\vec{n}$, will have to take another form where the effect of the azimuthal angle is accounted for. The heat flux will become

$$\vec{q} = -k\nabla T = -k\left[\vec{i}_r \frac{\partial}{\partial r} + \vec{i}_\theta \frac{1}{R}\frac{\partial}{\partial \theta}\right] T \tag{2.25}$$

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[5] In fact this calculation can be carried out at either the inner or the outer wall. The results obtained at any of the two cases are the same. But it is obvious that the calculation at the outer wall, the unperturbed wall, will be easier than it is at the inner wall, as for the outer wall case we do not have to redefine the unit outward normal which is, in this case, simply defined as $\vec{n} = \vec{i}_r$. However, in this document we chose to work with the more complicated case to learn how the calculations and the expansion will go.

and since the unit outward normal is given by

$$\vec{n} = \frac{\nabla f}{|\nabla f|} \tag{2.26}$$

where

$$f = r - R(\theta, \varepsilon) \tag{2.27}$$

and $f = 0$ defines the surface of the inner cylinder. Therefore,

$$\nabla f = \vec{i}_r - \vec{i}_\theta \frac{R'}{r} \tag{2.28}$$

where

$$R' = \frac{\partial R(\theta)}{\partial \theta} \tag{2.29}$$

and where at the surface ∇f becomes

$$\nabla f = \vec{i}_r - \vec{i}_\theta \frac{R'}{R} \tag{2.30}$$

Thus,

$$|\nabla f| = \sqrt{1 + \left(\frac{R'}{R}\right)^2} \tag{2.31}$$

Therefore, the unit outward normal is

$$\vec{n} = \frac{\vec{i}_r - \vec{i}_\theta \frac{R'}{R}}{\sqrt{1 + \left(\frac{R'}{R}\right)^2}} \tag{2.32}$$

and where dA is given by

$$dA = \sqrt{R'^2 + R^2}\, d\theta \tag{2.33}$$

By substituting the values from Equations (2.25), (2.32) and (2.33) into Equation (2.1), the total heat-transfer rate per unit length is turn to be

$$Q=-k\int_0^{2\pi}\sqrt{R'^2+R^2}\cdot\frac{\vec{i}_r-\vec{i}_\theta\dfrac{R'}{R}}{\sqrt{1+\left(\dfrac{R'}{R}\right)^2}}\cdot\left[\vec{i}_r\frac{\partial T}{\partial r}+\vec{i}_\theta\frac{1}{R}\frac{\partial T}{\partial\theta}\right]d\theta \tag{2.34}$$

Equation (2.34) can be simplified and written in the following form

$$Q=-k\left[\int_0^{2\pi}R\frac{\partial T}{\partial r}d\theta-\int_0^{2\pi}\frac{R'}{R}\frac{\partial T}{\partial\theta}d\theta\right] \tag{2.35}$$

whereupon substituting the expansion for the temperature at $r=R(\theta,\varepsilon)$, which gives

$$\begin{aligned}T(R,\theta)={}&T_o(R_o,\theta_o)+\varepsilon\left(T_1(R_o,\theta_o)+R_1(\theta_o)\frac{dT_o}{dr_o}(R_o)\right)\\&+\frac{1}{2}\varepsilon^2\left(\begin{aligned}&T_2(R_o,\theta_o)+2R_1(\theta_o)\frac{dT_1}{dr_o}(R_o,\theta_o)\\&+R_1^2(\theta_o)\frac{d^2T_o}{dr_o^2}(R_o)+R_2(\theta_o)\frac{dT_o}{dr_o}(R_o)\end{aligned}\right)+\ldots\end{aligned} \tag{2.36}$$

into Equation (2.35) using Equation (2.16), we can write Q in an expanded form. To simplify this step we will workout the first term on the right-hand side of Equation (2.35) then we will workout the second term. The first term will turn out to be

$$\begin{aligned}\int_0^{2\pi}R\frac{\partial T}{\partial r}d\theta={}&\int_0^{2\pi}\left(R_o+\varepsilon R_1+\frac{1}{2}\varepsilon^2R_2+\ldots\right)\cdot\left(\frac{\partial T_o}{\partial r_o}+\varepsilon\left[\frac{\partial T_1}{\partial r_o}+R_1\frac{\partial^2T_o}{\partial r_o^2}\right]\right.\\&\left.+\frac{1}{2}\varepsilon^2\left[\frac{\partial T_2}{\partial r_o}+2R_1\frac{\partial^2T_1}{\partial r_o^2}+R_1^2\frac{\partial^3T_o}{\partial r_o^3}+R_2\frac{\partial^2T_o}{\partial r_o^2}\right]+\ldots\right)d\theta_o\end{aligned} \tag{2.37}$$

and to the second order of ε it is

$$\begin{aligned}\int_0^{2\pi} R\frac{\partial T}{\partial r}d\theta = \int_0^{2\pi} R_o\frac{\partial T_o}{\partial r_o}d\theta_o + \varepsilon\left[\int_0^{2\pi} R_1\frac{\partial T_o}{\partial r_o}d\theta_o + \int_0^{2\pi} R_o\frac{\partial T_1}{\partial r_o}d\theta_o + \int_0^{2\pi} R_oR_1\frac{\partial^2 T_o}{\partial r_o^2}d\theta_o\right] \\ +\frac{1}{2}\varepsilon^2\left[\int_0^{2\pi} R_o\frac{\partial T_2}{\partial r_o}d\theta_o + \int_0^{2\pi} 2R_oR_1\frac{\partial^2 T_1}{\partial r_o^2}d\theta_o\right. \\ + \int_0^{2\pi} R_oR_1^2\frac{\partial^3 T_o}{\partial r_o^3}d\theta_o + \int_0^{2\pi} R_oR_2\frac{\partial^2 T_o}{\partial r_o^2}d\theta_o + \int_0^{2\pi} 2R_1\frac{\partial T_1}{\partial r_o}d\theta_o \\ \left. + \int_0^{2\pi} 2R_1^2\frac{\partial^2 T_o}{\partial r_o^2}d\theta_o + \int_0^{2\pi} R_2\frac{\partial T_o}{\partial r_o}d\theta_o\right]\end{aligned} \tag{2.38}$$

The second term will give

$$\int_0^{2\pi}\frac{R'}{R}\frac{\partial T}{\partial \theta}d\theta = \int_0^{2\pi}\frac{\left(\varepsilon R_1' + \frac{\varepsilon^2}{2}R_2' + \ldots\right)}{\left(R_o + \varepsilon R_1 + \frac{\varepsilon^2}{2}R_2 + \ldots\right)}\left[\varepsilon\frac{\partial T_1}{\partial \theta_o} + \frac{\varepsilon^2}{2}\left(\frac{\partial T_2}{\partial \theta_o} + 2R_1\frac{\partial^2 T_1}{\partial r_o \partial \theta_o}\right) + \ldots\right]d\theta_o \tag{2.39}$$

and to the second order of ε it is

$$\int_0^{2\pi}\frac{R'}{R}\frac{\partial T}{\partial \theta}d\theta = \int_0^{2\pi}\frac{(\varepsilon R_1')}{(R_o)}\left[\varepsilon\frac{\partial T_1}{\partial \theta_o}\right]d\theta_o \tag{2.40}$$

Then Q is just the difference between Equations (2.38) and (2.40), and it is

$$\begin{aligned}Q = \underbrace{-k\int_0^{2\pi} R_o\frac{\partial T_o}{\partial r_o}d\theta_o}_{Q_o} + \underbrace{\varepsilon(-k)\left[\int_0^{2\pi} R_1\frac{\partial T_o}{\partial r_o}d\theta_o + \int_0^{2\pi} R_o\frac{\partial T_1}{\partial r_o}d\theta_o + \int_0^{2\pi} R_oR_1\frac{\partial^2 T_o}{\partial r_o^2}d\theta_o\right]}_{Q_1} \\ \underbrace{+\frac{1}{2}\varepsilon^2(-k)\left[\begin{array}{l} \int_0^{2\pi} R_o\frac{\partial T_2}{\partial r_o}d\theta_o + \int_0^{2\pi} 2R_oR_1\frac{\partial^2 T_1}{\partial r_o^2}d\theta_o + \int_0^{2\pi} R_oR_1^2\frac{\partial^3 T_o}{\partial r_o^3}d\theta_o \\ + \int_0^{2\pi} R_oR_2\frac{\partial^2 T_o}{\partial r_o^2}d\theta_o + \int_0^{2\pi} 2R_1\frac{\partial T_1}{\partial r_o}d\theta_o + \int_0^{2\pi} 2R_1^2\frac{\partial^2 T_o}{\partial r_o^2}d\theta_o \\ + \int_0^{2\pi} R_2\frac{\partial T_o}{\partial r_o}d\theta_o - \int_0^{2\pi} 2\frac{R_1'}{R_o}\frac{\partial T_1}{\partial \theta}d\theta_o \end{array}\right]}_{Q_2}\end{aligned} \tag{2.41}$$

or Q can be written as

$$Q = Q_o + \varepsilon Q_1 + \frac{1}{2}\varepsilon^2 Q_2 + \ldots \tag{2.42}$$

where Q_o is the total heat-transfer rate for the symmetric case (*i.e.*, the reference case that we solved earlier), and Q_1 and Q_2 are the corrections terms because of the displacement of the inner cylinder.

Now, to solve for Q we must first solve for T_o, T_1 and T_2 from the heat-conduction equation where the effect of the azimuthal angle is accounted for. This is

$$0 = k\nabla^2 T = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]T \ , \quad \begin{cases} R(\theta,\varepsilon) < r < \kappa R_{in} \\ 0 < \theta < 2\pi \end{cases} \tag{2.43}$$

To achieve this, let us substitute the domain temperature expansion, which is given by Equation (2.22) into the domain equation (*i.e.*, Equation (2.43)). This gives

$$0 = k\left[\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o\frac{\partial}{\partial r_o}\right) + \frac{1}{r_o^2}\frac{\partial^2}{\partial\theta^2}\right]\cdot\left[T_o + \varepsilon T_1 + \frac{1}{2}\varepsilon^2 T_2\right] \tag{2.44}$$

and on requiring the coefficient of each power of ε to vanish, Equation (2.44) leads to the following set of linear differential equations. First at order ε^o, which is the symmetric case, the domain equation is

$$0 = \left[\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o\frac{\partial}{\partial r_o}\right)\right]T_o \tag{2.45}$$

where

$$T_o\left(r_o = \kappa R_{in}\right) = T_C \tag{2.46}$$

and

$$T_o\left(r_o = R_{in}\right) = T_H \tag{2.47}$$

This is the reference case, which was solved earlier and T_o was given by Equation (2.23), namely

$$T_o = \frac{(T_H - T_C)}{\ln(\kappa)} \ln\left(\frac{R_{in}}{r_o}\right) + T_H$$

For order ε^1, the domain equation is

$$0 = \left[\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o \frac{\partial}{\partial r_o}\right) + \frac{1}{r_o^2}\frac{\partial^2}{\partial \theta_o^2}\right] T_1 \tag{2.48}$$

where

$$T_1(r_o = \kappa R_{in}) = 0 \tag{2.49}$$

and

$$T_1(r_o = R_{in}) = -R_1 \left.\frac{dT_o}{dr_o}\right|_{r_o = R_{in}} \tag{2.50}$$

For order ε^2, the domain equation is

$$0 = \left[\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o \frac{\partial}{\partial r_o}\right) + \frac{1}{r_o^2}\frac{\partial^2}{\partial \theta_o^2}\right] T_2 \tag{2.51}$$

where

$$T_2(r_o = \kappa R_{in}) = 0 \tag{2.52}$$

and

$$T_2(r_o = R_{in}) = -2R_1 \left.\frac{dT_1}{dr_o}\right|_{r_o = R_{in}} - R_1^2 \left.\frac{d^2 T_o}{dr_o^2}\right|_{r_o = R_{in}} - R_2 \left.\frac{dT_o}{dr_o}\right|_{r_o = R_{in}} \tag{2.53}$$

To explain where Equations (2.50) and (2.53) come from[6], notice that the expansion of T at $r = R(\theta,\varepsilon)$, is given by Equation (2.36), namely by

$$T(R,\theta) = T_o(R_o,\theta_o) + \varepsilon\left(T_1(R_o,\theta_o) + R_1(\theta_o)\frac{dT_o}{dr_o}(R_o)\right) + \frac{1}{2}\varepsilon^2\left(\begin{array}{l} T_2(R_o,\theta_o) + 2R_1(\theta_o)\dfrac{dT_1}{dr_o}(R_o,\theta_o) \\ +R_1^2(\theta_o)\dfrac{d^2T_o}{dr_o^2}(R_o) + R_2(\theta_o)\dfrac{dT_o}{dr_o}(R_o) \end{array}\right) + \ldots$$

and that $T(R,\theta)$ must vanish for all values of ε.

Now, because of the form of R_1, the solution to the T_1 problem (*i.e.*, Equation (2.48)), can be written as

$$T_1 = \left(A_1 r_o + \frac{B_1}{r_o}\right)\cos\theta_o \tag{2.54}$$

where A_1 and B_1 must satisfy

$$A_1 \kappa R_{in} + \frac{B_1}{\kappa R_{in}} = 0 \tag{2.55}$$

and

$$A_1 R_{in} + \frac{B_1}{R_{in}} = -\left.\frac{dT_o}{dr_o}\right|_{r_o=R_{in}} \tag{2.56}$$

By solving Equations (2.55) and (2.56) for A_1 and B_1, T_1 is found to be

$$T_1 = \cos\theta_o \frac{1}{(\kappa^2-1)}\left[\frac{r_o}{R_{in}} - \frac{\kappa^2 R_{in}}{r_o}\right]\left.\frac{dT_o}{dr_o}\right|_{r_o=R_{in}} \tag{2.57}$$

and by this, there obtains

---

[6] Note that the perturbed boundary conditions at the inner wall are no longer isothermal because they try to simulate isothermal by putting a condition other than isothermal on the reference case.

$$\frac{dT_1}{dr_o}(r_o = R_{in}) = \cos\theta_o \frac{(\kappa^2+1)}{R_{in}(\kappa^2-1)} \left.\frac{dT_o}{dr_o}\right|_{r_o=R_{in}} \tag{2.58}$$

$$\frac{d^2T_1}{dr_o^2}(r_o = R_{in}) = \cos\theta_o \frac{-2\frac{\kappa^2}{R_{in}^2}}{(\kappa^2-1)} \left.\frac{dT_o}{dr_o}\right|_{r_o=R_{in}} \tag{2.59}$$

and

$$\frac{dT_1}{d\theta_o}(r_o = R_{in}) = \sin\theta_o \left.\frac{dT_o}{dr_o}\right|_{r_o=R_{in}} \tag{2.60}$$

which are needed in the T_2 problem.

To solve the T_2 problem (*i.e.*, Equation (2.51)), first substitute Equations (2.8), (2.9) and (2.58) into Equation (2.53) then expand the right-hand side of Equation (2.53) in terms of the functions[7] 1, $\cos\theta_o, \cos 2\theta_o$. Doing this Equation (2.53) reduces to

$$T_2(r_o = R_{in}) = \frac{-1}{R_{in}(\kappa^2-1)}\left[2 + (\kappa^2+1)\cos 2\theta_o\right] \left.\frac{dT_o}{dr_o}\right|_{r_o=R_{in}} \tag{2.61}$$

Then, from Equations (2.52) and (2.61), the solution to the T_2 problem can be written as

$$T_2 = A_o + B_o \ln(r_o) + \left(A_2 r_o^2 + \frac{B_2}{r_o^2}\right)\cos 2\theta_o \tag{2.62}$$

where A_o and B_o must satisfy

$$A_o + B_o \ln(\kappa R_{in}) = 0 \tag{2.63}$$

and

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[7] Substitute for $\cos^2\theta_o = \frac{1}{2} + \frac{1}{2}\cos 2\theta_o$ and for $\sin^2\theta_o = \frac{1}{2} - \frac{1}{2}\cos 2\theta_o$.

$$A_o + B_o \ln\left(R_{in}\right) = \frac{-2}{R_{in}\left(\kappa^2 - 1\right)} \left.\frac{dT_o}{dr_o}\right|_{r_o = R_{in}} \tag{2.64}$$

and where A_2 and B_2 must satisfy

$$A_2 \kappa^2 R_{in}^2 + \frac{B_2}{\kappa^2 R_{in}^2} = 0 \tag{2.65}$$

and

$$A_2 R_{in}^2 + \frac{B_2}{R_{in}^2} = \frac{-\left(\kappa^2 + 1\right)}{R_{in}\left(\kappa^2 - 1\right)} \left.\frac{dT_o}{dr_o}\right|_{r_o = R_{in}} \tag{2.66}$$

By solving Equations (2.63) and (2.64) for A_o and B_o, and Equations (2.65) and (2.66) for A_2 and B_2, T_2 is found to be

$$T_2 = -\left[\frac{2}{R_{in}\left(\kappa^2 - 1\right)\ln\left(\kappa\right)} \ln\left(\frac{\kappa R_{in}}{r_o}\right) + \frac{R_{in}\left(\kappa^2 + 1\right)}{\left(\kappa^2 - 1\right)\left(\kappa^4 - 1\right)} \left(\frac{\kappa^4}{r_o^2} - \frac{r_o^2}{R_{in}^4}\right) \cos 2\theta_o\right] \left.\frac{dT_o}{dr_o}\right|_{r_o = R_{in}} \tag{2.67}$$

and

$$\frac{dT_2}{dr_o} = \left[\frac{2}{R_{in}\left(\kappa^2 - 1\right)\ln\left(\kappa\right) r_o} + \frac{2R_{in}\left(\kappa^2 + 1\right)}{\left(\kappa^2 - 1\right)\left(\kappa^4 - 1\right)} \left(\frac{\kappa^4}{r_o^3} + \frac{r_o}{R_{in}^4}\right) \cos 2\theta_o\right] \left.\frac{dT_o}{dr_o}\right|_{r_o = R_{in}} \tag{2.68}$$

The aim now is to obtain the total heat-transfer rate for this case, which is defined earlier by Equation (2.41), that is

$$Q=\underbrace{-k\int_0^{2\pi}R_o\frac{\partial T_o}{\partial r_o}d\theta_o}_{Q_o}+\underbrace{\varepsilon(-k)\left[\int_0^{2\pi}R_1\frac{\partial T_o}{\partial r_o}d\theta_o+\int_0^{2\pi}R_o\frac{\partial T_1}{\partial r_o}d\theta_o+\int_0^{2\pi}R_oR_1\frac{\partial^2 T_o}{\partial r_o^2}d\theta_o\right]}_{Q_1}$$

$$\underbrace{+\frac{1}{2}\varepsilon^2(-k)\left[\begin{aligned}&\int_0^{2\pi}R_o\frac{\partial T_2}{\partial r_o}d\theta_o+\int_0^{2\pi}2R_oR_1\frac{\partial^2 T_1}{\partial r_o^2}d\theta_o+\int_0^{2\pi}R_oR_1^2\frac{\partial^3 T_o}{\partial r_o^3}d\theta_o\\&+\int_0^{2\pi}R_oR_2\frac{\partial^2 T_o}{\partial r_o^2}d\theta_o+\int_0^{2\pi}2R_1\frac{\partial T_1}{\partial r_o}d\theta_o+\int_0^{2\pi}2R_1^2\frac{\partial^2 T_o}{\partial r_o^2}d\theta_o\\&+\int_0^{2\pi}R_2\frac{\partial T_o}{\partial r_o}d\theta_o-\int_0^{2\pi}2\frac{R_1'}{R_o}\frac{\partial T_1}{\partial\theta}d\theta_o\end{aligned}\right]}_{Q_2}$$

In this equation, Q_o is the total heat-transfer rate for the reference case (*i.e.*, the symmetric case), which is given by Equation (2.24) as

$$Q_o=2\pi k\frac{(T_H-T_C)}{\ln(\kappa)}>0$$

Q_1 is the coefficient of the first perturbation term (*i.e.*, ε^1), found as

$$\begin{aligned}Q_1=-k\Bigg[&-\int_0^{2\pi}-\cos\theta_o\frac{(T_H-T_C)}{R_{in}\ln(\kappa)}d\theta_o\\&+\int_0^{2\pi}\cos\theta_o\left(\frac{\kappa^2+1}{\kappa^2-1}\right)\left.\frac{dT_o}{dr_o}\right|_{r_o=R_{in}}d\theta_o\\&+\int_0^{2\pi}\cos\theta_o\frac{(T_H-T_C)}{R_{in}\ln(\kappa)}d\theta_o-\Bigg]\end{aligned}\tag{2.69}$$

and since

$$\int_0^{2\pi}\cos\theta_o d\theta=\sin\theta_o\Big|_0^{2\pi}=0\tag{2.70}$$

we can see that

$$Q_1=0\tag{2.71}$$

This means Q_1 appearing in Equation (2.41) has no effect on the total Q. This is an expected result because we know intuitively that it should make no difference in the answer of this problem if the inner rod is shifted to the right (*i.e.*, $\varepsilon > 0$), or to the left (*i.e.*, $\varepsilon < 0$). Then the only possibility by which we can get the same answer, regardless of the sign of ε, is for Q_1 to be zero. By the same account, we expect all the odd Q's to be zero (*i.e.*, $Q_3 = 0,\ Q_5 = 0, \ldots$). Hence to learn the effect of a displacement we must go to second order in ε.

The value of Q_2 which is the coefficient of the second perturbation term (*i.e.*, ε^2), is equal to

$$Q_2 = k\frac{4\pi\left(T_H - T_C\right)}{R_{in}^2\left(\kappa^2 - 1\right)\left(\ln\left(\kappa\right)\right)^2} > 0 \tag{2.72}$$

Note that the value of Q_2 is always positive, since κ can only be greater than one.

Based on these results, and since $\varepsilon,\ Q_o$ and Q_2 are positive quantities, we can observe that the total heat-transfer rate Q for the distorted annulus ($Q = Q_o + \frac{1}{2}\varepsilon^2 Q_2$) is greater than that of the perfectly centered one by an amount equal to $\varepsilon^2 Q_2/2$.

Now, to get an idea about how much the total heat-transfer rate Q is affected by perturbing the inner rod, we need to substitute numerical values into Equation (2.24), namely

$$Q_o = 2\pi k\frac{\left(T_H - T_C\right)}{\ln\left(\kappa\right)}$$

and Equation (2.42), which can be written as

$$Q = 2\pi k \frac{(T_H - T_C)}{\ln(\kappa)} + \frac{1}{2}\varepsilon^2 k \frac{4\pi(T_H - T_C)}{R_{in}^2(\kappa^2 - 1)(\ln(\kappa))^2}$$

Then a comparison between the values obtained from each of these two equations can be made.

Several numerical values are examined for different values of ε, T_H, T_C and κ at which R_{in} is taken to be equal to 1. From this we find that the total heat-transfer rate for the distorted annulus, $Q_{perturbed}$, is greater than that of the perfectly centered one, $Q_{centered}$. The results obtained from two of these calculations at which $T_H = 100$ and $T_C = 10$ are illustrated in Table 2-1 and Table 2-2.

Table 2-1. Comparison of values $Q_{centered}$ and $Q_{perturbed}$ for several values of κ and at $\varepsilon = 0.1$.

κ	$Q_{centered}/k$ (T)	$Q_{perturbed}/k$ (T)	$\frac{Q_{perturbed}}{Q_{centered}}$	κ	$Q_{centered}/k$ (T)	$Q_{perturbed}/k$ (T)	$\frac{Q_{perturbed}}{Q_{centered}}$
1.5	1394.66	1422.18	1.01900	6	315.60	315.65	1.00016
2	815.820	819.750	1.00500	7	290.60	290.63	1.00011
3	514.720	515.310	1.00100	8	271.94	271.96	1.00007
4	407.910	408.110	1.00048	9	257.36	257.38	1.00005
5	351.350	351.450	1.00026	10	245.58	245.60	1.00004

Table 2-2. Comparison of values $Q_{centered}$ and $Q_{perturbed}$ for several values of κ and at $\varepsilon = 0.2$.

κ	$Q_{centered}/k$ (T)	$Q_{perturbed}/k$ (T)	$\frac{Q_{perturbed}}{Q_{centered}}$	κ	$Q_{centered}/k$ (T)	$Q_{perturbed}/k$ (T)	$\frac{Q_{perturbed}}{Q_{centered}}$
1.5	1394.66	1504.73	1.0790	6	315.60	315.81	1.0006
2	815.820	831.520	1.0190	7	290.60	290.73	1.0004
3	514.720	517.070	1.0046	8	271.94	272.02	1.0003
4	407.910	408.700	1.0019	9	257.36	257.42	1.0002
5	351.350	351.720	1.0010	10	245.58	245.63	1.0001

From the tabulated results we see that, as κ increases the values of the total heat-transfer rate in both the perfectly centered and the distorted annulus case decrease and become closer to each other. From this we understand that thick regions are poor heat conductors and they are less sensitive to small perturbation. Beside this, we can notice that as the displacement, ε , increases the total heat-transfer rate increases. This happens because by increasing ε the heat-transfer rate by conduction across the thin region increases and consequently the azimuthal gradient increases.

From the above we can conclude that, when the inner cylinder is perturbed as shown in (Figure 2-1B) it is clear that the heat-transfer rate can change only at order ε^2 and that the total heat-transfer rate increases. Therefore, the distorted annulus can tolerate more heat generation because it appears to be a better heat conductor than the perfectly centered one. Observe that this result is similar to the one expected from the picture argument.

Finally, it is important to state that, the same results and conclusions can be obtained if the calculation to obtain the total heat-transfer rate is carried out at the outer wall or even if the outer wall is displaced instead of the inner wall.

CHAPTER 3
THERMAL STABILITY IN A PLANAR GEOMETRY WITH HEAT BEING PRODUCED WITHIN THE DOMAIN

In this chapter we use a simple problem of a planar geometry to get an understanding of the tools required to accomplish our objective. Then what might be learned from this simple problem can be applied to the problems of this dissertation that are, of course, more complicated.

Now, before the calculations are presented and discussed, one might like to know what is going on from a physical point of view. To achieve this, consider Figure 3-1. It depicts a heat-producing material confined in the space between two parallel plane walls a distance $2L$ apart. Somewhere inside the heat producing material is a layer of non-heat producing material; its thermal conductivity is denoted k^C, a variable whose value can range from zero to infinity and its thickness is 2β. Heat is lost to the boundaries, which are held at a fixed temperature denoted T_w. The thermal conductivity of the material in the space is denoted by k, and ε denotes the distance by which the central layer is shifted from the center of the geometry. The heat source will be taken to depend linearly on the temperature of the heat-generating material and it will be of interest to predict the maximum temperature in the heat-producing region as a function of the parameters of the problem. A thermal explosion will be identified by an infinite maximum temperature which in turn translates into our inability to find a steady solution.

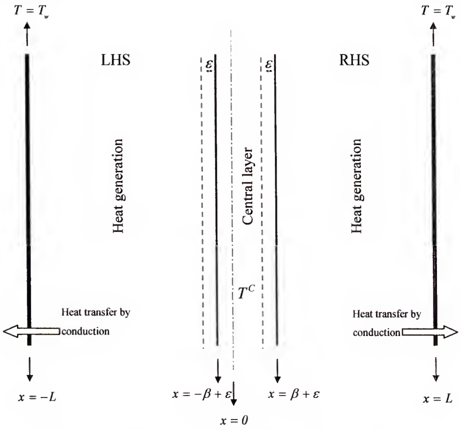

Figure 3-1. Planar geometry of with the heat-producing material.

Let us say that we are interested in knowing how the conditions for thermal explosion change as the position of the central layer is shifted and its thermal conductivity changes.

Now the heat released in the material in the space between the two parallel walls can be removed by conduction to the plane walls only if the temperature of the material is higher than that of the walls. To conduct the heat away, the more distant a point of the material is from the boundary the higher its temperature must rise. Knowing this, let us change the walls temperatures and positions until the heat generated in the material becomes greater than that conducted through the walls at which point thermal instability may occur.

Now, before the calculations for this simple problem are presented and discussed, let us try to predict what is happening at various positions and conditions of the central layer, such as those depicted in Figure 3-2B, C and D. Our prediction will be based on the solution of the easiest sub-case of the planar problem *where no central layer exists between the two parallel walls*, as depicted in Figure 3-2A. The solution of this heat transfer problem is presented in the following.

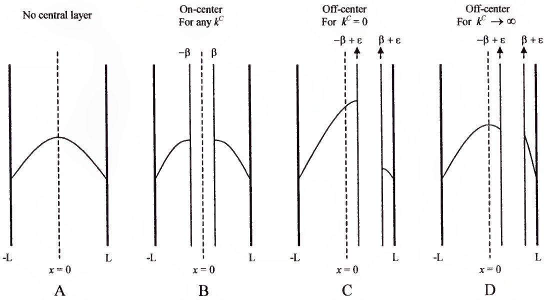

Figure 3-2. Geometries of the cases for which the prediction is made. A) No solid layer. B) Solid layer On-center. C) Adiabatic solid layer. D) Perfectly conductive solid layer.

The temperature rise in this problem, referring to Figure 3-3 for a definition of the variables, can be described by the thermal energy equation

$$\rho C_v \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + Q(T) \tag{3.1}$$

where ρ and C_v denote the density and the heat capacity of the heat-producing material, respectively. They are assumed to be constant through this work. T is the temperature field and $Q(T)$ denotes the rate at which heat is produced in the material per unit volume, assumed to be a linear function in T. The equation is subject to two boundary conditions viz.,

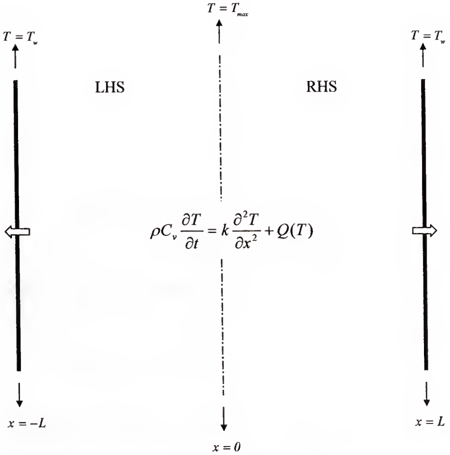
$T = T_{max}$
$T = T_w$
$T = T_w$
LHS
RHS
$\rho C_v \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + Q(T)$
$x = -L$
$x = 0$
$x = L$

Figure 3-3. Planar geometry when no central layer exists.

$$T(x = -L) = T_w \tag{3.2}$$

and

$$T(x = L) = T_w \tag{3.3}$$

Let $Q(T)$ be a linear function of T (*i.e.*, $Q(T) = Q_o + Q_1(T - T_w)$) and set $\dot{T} = T - T_w$. Based on Equation (3.1) $\dot{T}$ must satisfy

$$\rho C_v \frac{\partial \dot{T}}{\partial t} = k\frac{\partial^2 \dot{T}}{\partial x^2} + Q_0\left(1 + S\dot{T}\right) \tag{3.4}$$

where

$$S = \frac{Q_1}{Q_o}$$

which is the sensitivity of the heat generation to temperature. Equation (3.4) can be written in partly scaled form by defining $\dot{T} = \overline{T}\ T^*$ and $t = \overline{t}\ t^*$, where $\overline{T} = \frac{1}{S}$ and where $\overline{t} = \frac{\rho C_v}{k} L^2$. Then the equation to be solved is

$$\frac{1}{\alpha}\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial x^2} + \delta^2 (1 + T^*)\ , \quad -L < x < L \tag{3.5}$$

where

$$\alpha = \frac{k\overline{t}}{\rho C_v} \text{ and } \delta^2 = \frac{SQ_o}{k}$$

The value of δ^2 indicates the strength of the heat source and the linear function $(1 + T^*)$ indicates how fast it increases as T^* increases.

For simplicity in nomenclature, let us replace the scaled time t^* and the scaled temperature T^* in Equation (3.5) by t and T, respectively, (*i.e.*, drop the $*$'s). Then the steady state temperature rise must satisfy

$$\frac{d^2 T}{dx^2} + \delta^2 (1 + T) = 0 \tag{3.6}$$

where

$$T(x = -L) = 0 \tag{3.7}$$

and

$$T(x = L) = 0 \tag{3.8}$$

The input variables of this problem are L, k and δ^2. The input variables L and k are held fixed. The only output variable, at steady state, is therefore T. It depends on x.

Equation (3.6) is a linear, non-homogenous, second order ODE that can be solved to give

$$T = A\cos\delta x + B\sin\delta x - 1\,,\quad -L < x < L \tag{3.9}$$

Now apply the two boundary conditions to find the constants A and B. This will give

$$A\cos\delta L - B\sin\delta L - 1 = 0\,,\ x = -L \tag{3.10}$$

and

$$A\cos\delta L + B\sin\delta L - 1 = 0\,,\ x = L \tag{3.11}$$

whence T for this sub-case of the planar problem can be written as

$$T = \frac{\cos\delta x}{\cos\delta L} - 1 \tag{3.12}$$

For this case, the temperature profile between the two walls is depicted in Figure 3-4.

The aim of this calculation is to predict the critical value of δ (*i.e.*, the value of δ beyond which no solution exists *i.e.*, where the output variable T is not defined). From Equation (3.12) one can see that T is not defined when the denominator of the first term on the right-hand side vanishes (*i.e.*, when $\cos\delta L = 0$), where δ is the only possible input variable and its value can be freely set. The critical values of δ at which T ceases to exist are termed $\delta_{critical}$.

From Equation (3.12) the critical value of δ is found to be[1]

$$\delta_{critical} = \frac{\pi}{2L} \tag{3.13}$$

Now, based on the calculations done for case A, let us predict how the critical points get shifted for the cases shown in Figure's 3-2B, C and D.

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[1] $\cos\delta L = 0$ for $\delta L = n\pi/2$, where $n = 1,3,5\ldots$, $\delta_{critical}$ will be the first of these values encountered in every δ from zero.

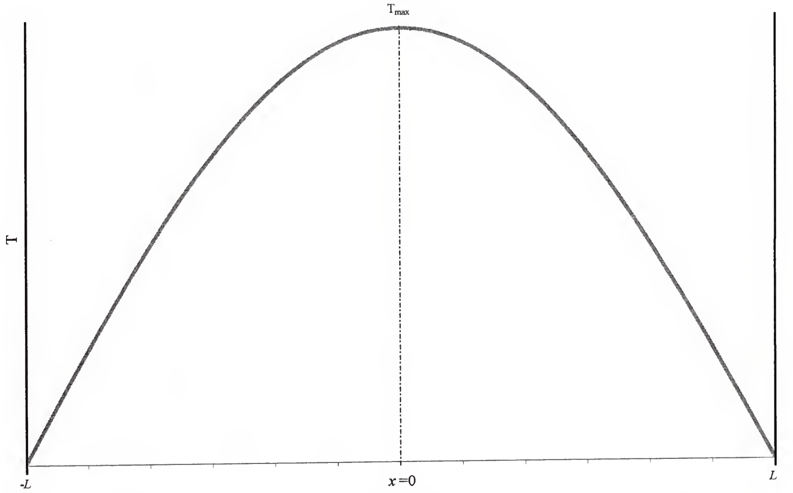

Figure 3-4. The temperature profile where no-central layer exists between the two parallel walls.

For case B the central layer is located in the center of the geometry and this case is symmetric. Because of this symmetry we can say that no heat is conducted through the central layer and therefore the value of k^C would have no effect. The temperature in the central layer will be uniform and it will be the highest temperature existing between the outside walls. By comparing this case with case A we can see that the only effect of having a central layer is to change the thickness of the heat-generating region. In other words, in case A the thickness is $(L-0)$ while in case B it is $(L-\beta)$. From this the critical point is expected to be

$$\delta_{critical} = \frac{\pi}{2(L-\beta)} \tag{3.14}$$

The critical value of δ in case B is greater than it is in case A. This might mean that, case B can stand a bigger S and/or smaller k and/or larger L than case A. This happens because of the fact that the thickness of the heat-generating region in case B is less than it is in case A. And consequently, case B has less quantity of the heat-generating material. From this we can conclude that thin regions of heat-generating material are thermally more stable than thick regions.

Now let us return to our job of making predictions based on case A. Going on to case C the central layer is shifted a distance ε from the center of the geometry but no heat transfer is permitted through this layer as it is assumed that $k^C = 0$. This leads to two completely independent heat-generating regions as no heat can be conducted through the insulated layer. The temperature in each region is at its maximum at its adiabatic surface and each region acts like a case A problem at twice its thickness. Hence, the thickness of the thicker side determines the critical value of δ. In case A the thickness is $(L-0)$

while in case C two different distances must be taken into account, $(L-\beta+\varepsilon)$ for the thick region and $(L-\beta-\varepsilon)$ for the thin region, which represent the left and the right side, respectively. The predicted critical point would then be expected to be the lower of

$$\delta_{critical}=\frac{\pi}{2(L-\beta+\varepsilon)} \tag{3.15}$$

and

$$\delta_{critical}=\frac{\pi}{2(L-\beta-\varepsilon)} \tag{3.16}$$

Equation (3.15) must be controlling because its prediction is the lower one.

Finally, in case D the central layer is shifted a distance ε from the center of the geometry but now the heat-generating regions remain in perfect contact as $k^C \rightarrow \infty$. Based on this, one can deal with this case as if no layer exists between the two parallel walls. Then the values of ε will have no effect on the results and only β must be considered. The temperature profile would then be similar to that of case A except that the central layer will cut this curve into two pieces, separating the two pieces by the thickness of the layer, 2β. Across the central layer the temperature is uniform. By comparing this case with case A we can see that the effect is to change the distance from the geometric center to the walls. The distance in this case, case D, is $(L-\beta)$. From this the expected critical point would be thought to be

$$\delta_{critical}=\frac{\pi}{2(L-\beta)} \tag{3.17}$$

The predicted critical points for cases B, C and D will be shown to be correct from the calculations given in the following section.

3.1 The Parallel Plane Wall Problem

This problem is depicted in Figure 3-1 and described in full earlier. The temperature rise T at the left-hand side must satisfy

$$\rho C_v \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + Q(T) \tag{3.18}$$

at the right-hand side it must satisfy

$$\rho C_v \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + Q(T) \tag{3.19}$$

and in the central layer it must satisfy

$$\rho^C C_v^C \frac{\partial T^C}{\partial t} = k^c \frac{\partial^2 T^C}{\partial x^2} \tag{3.20}$$

where $Q(T)$ denotes the rate at which heat is produced in the material per unit volume. It is positive and it increases as T increases. Some notation is illustrated in Figure 3-5.

The boundary conditions are

$$T(x=-L) = T_w \tag{3.21}$$

$$T(x=-\beta+\varepsilon) = T^C(x=-\beta+\varepsilon) \tag{3.22}$$

$$T(x=L) = T_w \tag{3.23}$$

and

$$T(x=\beta+\varepsilon) = T^C(x=\beta+\varepsilon) \tag{3.24}$$

with two more conditions at the interface, namely

$$k\frac{dT}{dx} = k^C \frac{dT^C}{dx} \text{ at } x=-\beta+\varepsilon \tag{3.25}$$

and

$$k\frac{dT}{dx} = k^C \frac{dT^C}{dx} \text{ at } x = \beta + \varepsilon \tag{3.26}$$

The initial condition is

$$T = f(x) \tag{3.27}$$

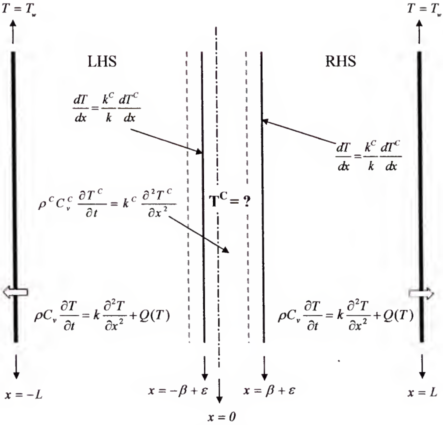
$T = T_w$
$T = T_w$
LHS
RHS
$\frac{dT}{dx} = \frac{k^C}{k}\frac{dT^C}{dx}$
$\frac{dT}{dx} = \frac{k^C}{k}\frac{dT^C}{dx}$
$\rho^C C_v^C \frac{\partial T^C}{\partial t} = k^C \frac{\partial^2 T^C}{\partial x^2}$
$T^C = ?$
$\rho C_v \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + Q(T)$
$\rho C_v \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + Q(T)$
$x = -L$
$x = -\beta + \varepsilon$
$x = \beta + \varepsilon$
$x = L$
$x = 0$

Figure 3-5. Planar geometry with the domain equations.

Again, let $Q(T)$ be a linear function of T (*i.e.*, $Q(T) = Q_o + Q_1(T - T_w)$) and again set $\dot{T} = T - T_w$. Next rewrite Equations (3.18), (3.19) and (3.20) in partly scaled form by defining $\dot{T} = \overline{T}\ T^*$ and $t = \overline{t}\ t^*$, where $\overline{T} = \frac{1}{S}$ and where $\overline{t} = \frac{\rho C_v}{k}L^2$. Then the equations to be satisfied are

$$\frac{1}{\alpha}\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial x^2} + \delta^2(1 + T^*)\,, \quad -L < x < -\beta + \varepsilon \tag{3.28}$$

$$\frac{1}{\alpha}\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial x^2} + \delta^2(1+T^*)\ , \quad \beta+\varepsilon < x < L \tag{3.29}$$

and

$$\frac{1}{\alpha^C}\frac{\partial T^{*C}}{\partial t^*} = \frac{\partial^2 T^{*C}}{\partial x^2}\ , \quad -\beta+\varepsilon < x < \beta+\varepsilon \tag{3.30}$$

where

$$\alpha = \frac{k\bar{t}}{\rho C_v}, \quad \alpha^C = \frac{k^C\bar{t}}{\rho^C C_v^C}, \quad \delta^2 = \frac{SQ_o}{k} \quad \text{and } S = \frac{Q_1}{Q_o}$$

As before the value of δ^2 indicates the strength of the heat source and the linear function $(1+T^*)$ indicates how fast it increases as T^* increases. Again, for simplicity in notation, drop the * 's.

The input variables for this problem are L, β, ε, k, k^C and δ^2. All the input variables are held fixed except δ^2. The only output variable, at steady state, is therefore T. It depends on x.

By looking at Equations (3.28), (3.29) and (3.30) one can see that we are fortunate as they can be solved at steady state conditions. The steady state solutions to these equations are derived in the following section.

3.2 Observations on the Steady State Solutions

From Equations (3.28), (3.29) and (3.30) the steady state temperature rise T_{steady} must satisfy

$$\frac{d^2 T_{steady}}{dx^2} + \delta^2(1+T_{steady}) = 0\ , \quad -L < x < -\beta+\varepsilon \tag{3.31}$$

$$\frac{d^2 T_{steady}}{dx^2} + \delta^2(1+T_{steady}) = 0\ , \quad \beta+\varepsilon < x < L \tag{3.32}$$

and

$$\frac{d^2T^C_{steady}}{dx^2}=0\ ,\quad -\beta+\varepsilon<x<\beta+\varepsilon \tag{3.33}$$

where at the boundaries and at the surfaces separating the regions we must have

$$T_{steady}(x=-L)=0 \tag{3.34}$$

$$T_{steady}(x=-\beta+\varepsilon)=T^C_{steady}(x=-\beta+\varepsilon) \tag{3.35}$$

$$T_{steady}(x=L)=0 \tag{3.36}$$

and

$$T_{steady}(x=\beta+\varepsilon)=T^C_{steady}(x=\beta+\varepsilon) \tag{3.37}$$

as well as

$$k\frac{dT_{steady}}{dx}=k^C\frac{dT^C_{steady}}{dx}\ ,\quad x=-\beta+\varepsilon \tag{3.38}$$

and

$$k\frac{dT_{steady}}{dx}=k^C\frac{dT^C_{steady}}{dx}\ ,\quad x=\beta+\varepsilon \tag{3.39}$$

Equations (3.31) and (3.32) are linear, non-homogenous, second order ODEs which can be solved easily to give

$$T_{steady}=A\cos\delta x+B\sin\delta x-1\ ,\quad -L<x<-\beta+\varepsilon \tag{3.40}$$

and

$$T_{steady}=C\cos\delta x+D\sin\delta x-1\ ,\quad \beta+\varepsilon<x<L \tag{3.41}$$

Equation (3.33) is a linear, homogenous, second order ODE that can be solved to give

$$T^C_{steady}=E+Fx\ ,\quad -\beta+\varepsilon<x<\beta+\varepsilon \tag{3.42}$$

Now we substitute these results into the six boundary conditions to determine A, B, C, D, E and F. This will give

$$A\cos\delta L - B\sin\delta L - 1 = 0\ ,\ x = -L \tag{3.43}$$

$$A\cos\delta(\beta-\varepsilon) - B\sin\delta(\beta-\varepsilon) - 1 - E + F\beta - F\varepsilon = 0\ ,\ x = -\beta+\varepsilon \tag{3.44}$$

$$C\cos\delta L + D\sin\delta L - 1 = 0\ ,\ x = L \tag{3.45}$$

$$C\cos\delta(\beta+\varepsilon) + D\sin\delta(\beta+\varepsilon) - 1 - E - F\beta - F\varepsilon = 0\ ,\ x = \beta+\varepsilon \tag{3.46}$$

$$A\delta\sin\delta(\beta-\varepsilon) + B\delta\cos\delta(\beta-\varepsilon) - \frac{k^C}{k}F = 0\ ,\ x = -\beta+\varepsilon \tag{3.47}$$

and

$$-C\delta\sin\delta(\beta+\varepsilon) + D\delta\cos\delta(\beta+\varepsilon) - \frac{k^C}{k}F = 0\ ,\ x = \beta+\varepsilon \tag{3.48}$$

Then, solve Equations (3.43) and (3.47) to find the constants A and B in terms of F and solve Equations (3.45) and (3.48) to find the constants C and D in terms of F. The results are

$$A = \frac{\delta\cos\delta\left(\beta-\varepsilon\right) + \frac{k^C}{k}F\sin\delta L}{\delta\cos\delta\left(L-\beta+\varepsilon\right)} \tag{3.49}$$

$$B = \frac{\frac{k^C}{k}F\cos\delta L - \delta\sin\delta\left(\beta-\varepsilon\right)}{\delta\cos\delta\left(L-\beta+\varepsilon\right)} \tag{3.50}$$

$$C = \frac{\delta\cos\delta\left(\beta+\varepsilon\right) - \frac{k^C}{k}F\sin\delta L}{\delta\cos\delta\left(L-\beta-\varepsilon\right)} \tag{3.51}$$

and

$$D = \frac{\frac{k^C}{k}F\cos\delta L + \delta\sin\delta\left(\beta+\varepsilon\right)}{\delta\cos\delta\left(L-\beta-\varepsilon\right)} \tag{3.52}$$

Next, substitute A and B given by Equations (3.49) and (3.50) into Equation (3.44) which gives

$$\frac{1}{\cos\delta(L-\beta+\varepsilon)}+\frac{k^C}{\delta k}F\tan\delta(L-\beta+\varepsilon)-1-E+F\beta-F\varepsilon=0 \tag{3.53}$$

and substitute C and D given by Equations (3.51) and (3.52) into Equation (3.46) which gives

$$\frac{1}{\cos\delta(L-\beta-\varepsilon)}-\frac{k^C}{\delta k}F\tan\delta(L-\beta-\varepsilon)-1-E-F\beta-F\varepsilon=0 \tag{3.54}$$

Then, to find F in terms of $L,\ \beta,\ \varepsilon$ and k^C/k, subtract Equation (3.53) from Equation (3.54), whence

$$\begin{aligned} &F\left[\frac{k^C}{\delta k}\left[\tan\delta(L-\beta+\varepsilon)+\tan\delta(L-\beta-\varepsilon)\right]+2\beta\right] \\ &\qquad=\left[\frac{1}{\cos\delta(L-\beta-\varepsilon)}-\frac{1}{\cos\delta(L-\beta+\varepsilon)}\right] \end{aligned} \tag{3.55}$$

To find the critical value of δ as a function of $L,\ \beta,\ \varepsilon$ and k^C/k we seek values of δ where the solution does not exist (*i.e.*, where F is not defined). From Equation (3.55) we see that F is not defined at one of the following two conditions. The first condition is when the term multiplying it on the left-hand side vanishes. And the second condition is when the term on the right-hand side of Equation (3.55) goes to infinity. When the first condition is considered, the critical value of δ must satisfy

$$\left[\frac{k^C}{\delta k}\left[\tan\delta(L-\beta+\varepsilon)+\tan\delta(L-\beta-\varepsilon)\right]+2\beta\right]=0 \tag{3.56}$$

Now Equation (3.56) can be arranged as

$$\left[\tan\delta(L-\beta+\varepsilon)+\tan\delta(L-\beta-\varepsilon)\right]=-\frac{2\beta\delta k}{k^C} \tag{3.57}$$

and the critical value of δ can be found by substituting numerical values into this equation and then the left hand and the right hand sides of this equation need to be drawn as graphs *vs.* δ. The points of intersection are the possible critical values of δ.

Therefore, several numerical values of L, β, ε, k and k^C are examined.[2] Then, to find the critical value of δ, let us begin by plotting the left-hand side of Equation (3.57) *vs.* δ. This is made to see how the graph of the left-hand side will look like and to study the effect of change in ε on its shape. Thus, the left-hand side is drawn for two different values of ε and the results are depicted in Figures 3-6 and 3-7.

From these two figures we can note that, as δ increases the left-hand side term will form the usual tangent curve. Also we see that, as the value of ε increases the tangent curve becomes less sharper.

Finally, both the left-hand side and the right-hand side of Equation (3.57) are considered and drawn as graphs *vs.* δ for two different values of ε as depicted in Figure 3-8 and Figure 3-9. The points of intersection are the possible critical values of δ at the selected values of L, β, ε, k and k^C.

From Figure 3-8 and Figure 3-9 we can note that, as δ increases the left-hand side term of Equation (3.57) will form the usual tangent curve while the right-hand side of the equation will decrease linearly with δ. Moreover, by changing the values of k^C and ε a very important conclusion can be drawn. It is this: as k^C increases $\delta_{critical}$ increases and

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[2] The values of k and k^c used in this work and in these calculations cover a wide range of materials. The values of k used cover various types of materials such as solid metals, nonmetallic solids, nonmetallic liquids and liquid metals while the values of k^c used include solid metals and nonmetallic solids. The approximate ranges for the thermal conductivity of various classes of materials can be found in Deen, W.M., 1998, Analysis of Transport Phenomena, Oxford University Press, New York. The numerical values for the thermal conductivity of various materials can be found in the CRC Handbook of Chemistry and Physics, CRC Press, 70th Ed., Boca Raton, Florida.

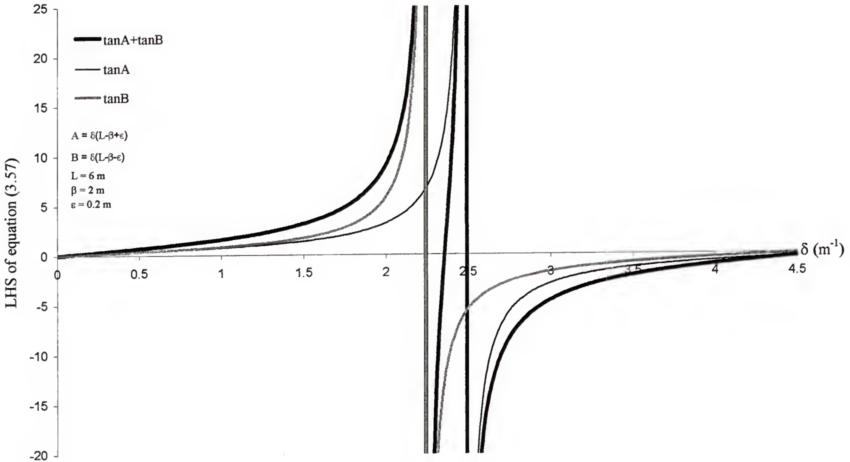

Figure 3-6. Shape of $\left[\tan\delta(L-\beta+\varepsilon)+\tan\delta(L-\beta-\varepsilon)\right]$ and δ at $\varepsilon = 0.2\,\text{m}$.

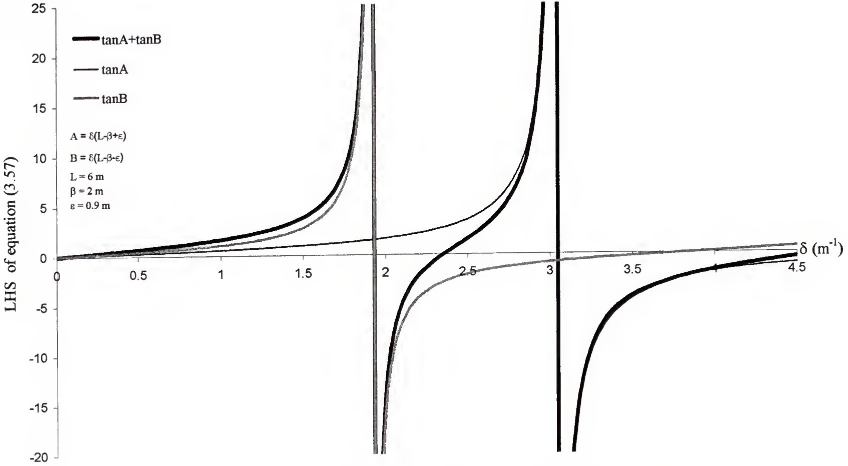

Figure 3-7. Shape of $\left[\tan\delta\left(L-\beta+\varepsilon\right)+\tan\delta\left(L-\beta-\varepsilon\right)\right]$ and δ at $\varepsilon = 0.9\,\text{m}$.

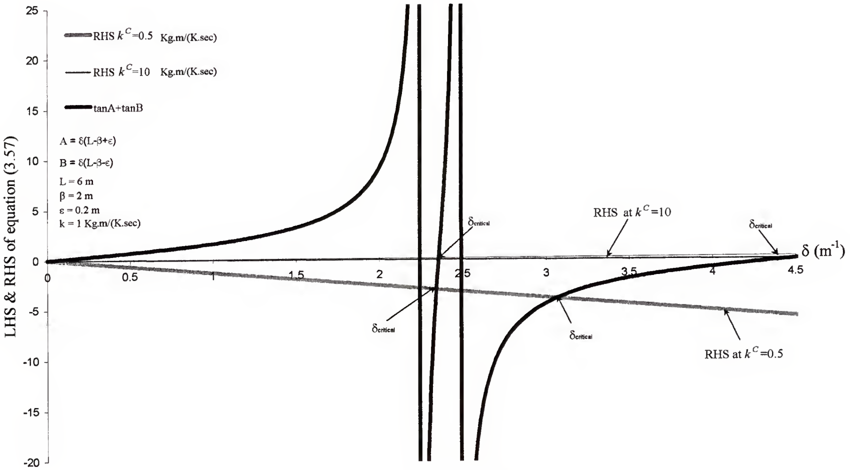

Figure 3-8. Critical value of δ at $\varepsilon = 0.2\,\mathrm{m}$ and different values of k^C.

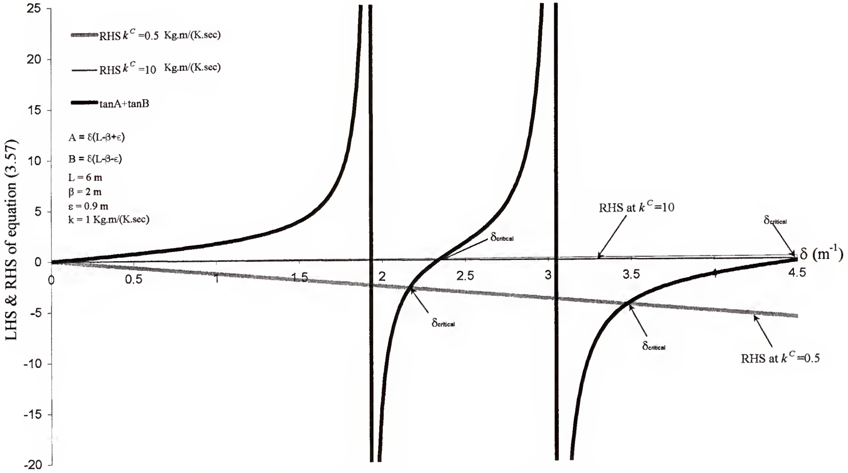

Figure 3-9. Critical value of δ at $\varepsilon = 0.9\,\text{m}$ and different values of k^C.

as ε increases $\delta_{critical}$ decreases. This might happen because of the fact that, as k^C increases more heat from the thick region can be conducted through the central layer toward the thin region and this can increase the thermal stability of the system. On the other hand, as ε increases the thick region becomes thicker and, therefore, it will generate more heat and, for such a case, the thermal stability of the system decreases.

When the second condition is considered (*i.e.*, when the term on the right-hand side of Equation (3.55) goes to infinity), the critical value of δ must satisfy

$$\left[\frac{\cos\delta(L-\beta+\varepsilon)-\cos\delta(L-\beta-\varepsilon)}{\cos\delta(L-\beta-\varepsilon)\cos\delta(L-\beta+\varepsilon)}\right]\rightarrow\infty \tag{3.58}$$

This mean that, either $\cos\delta(L-\beta-\varepsilon)=0$ or $\cos\delta(L-\beta+\varepsilon)=0$. Therefore, two critical points can be found. The critical point would then be expected to be the lower of

$$\delta_{critical}=\frac{\pi}{2(L-\beta+\varepsilon)} \tag{3.59}$$

and

$$\delta_{critical}=\frac{\pi}{2(L-\beta-\varepsilon)} \tag{3.60}$$

Equation (3.59) must be controlling because it is the lower one.

Now, two different limits of k^C can be investigated. First let k^C be *zero*. This means that no heat can transfer through the central layer and this will result in the formation of two completely independent heat-generating regions, one to the left of the insulator and the other to its right. Equation (3.55) will then turn into

$$F=\frac{1}{2\beta}\left[\frac{1}{\cos\delta(L-\beta-\varepsilon)}-\frac{1}{\cos\delta(L-\beta+\varepsilon)}\right] \tag{3.61}$$

which can be rearranged as

$$F = \frac{1}{2\beta}\left[\frac{\cos\delta(L-\beta+\varepsilon)-\cos\delta(L-\beta-\varepsilon)}{\cos\delta(L-\beta-\varepsilon)\cos\delta(L-\beta+\varepsilon)}\right] \tag{3.62}$$

Then, to find $\delta_{critical}$ the denominator on the right-hand side of Equation (3.62) must vanish, which means that

$$\cos\delta(L-\beta-\varepsilon)\cos\delta(L-\beta+\varepsilon)=0 \tag{3.63}$$

must hold, and based on this two critical points can be found. They are

$$\delta_{critical} = \frac{\pi}{2(L-\beta+\varepsilon)} \tag{3.64}$$

and

$$\delta_{critical} = \frac{\pi}{2(L-\beta-\varepsilon)} \tag{3.65}$$

These are the results predicted earlier. Equation (3.64) must be controlling because it gives the smaller critical value of δ.

Now let k^C be *infinitely large*. Again, the critical value of δ corresponds to not being able to find F, and for this to be the case the term multiplying F in Equation (3.55) must vanish (*i.e.*, Equation (3.56)),

$$\left[\frac{k^C}{\delta k}\left[\tan\delta(L-\beta+\varepsilon)+\tan\delta(L-\beta-\varepsilon)\right]+2\beta\right]=0$$

must hold. This can be rearranged as

$$\left[\tan\delta(L-\beta+\varepsilon)+\tan\delta(L-\beta-\varepsilon)\right] = -\frac{2\beta\delta k}{k^C} \tag{3.66}$$

where the right-hand side goes to zero as $k^C \rightarrow \infty$ (provided δ does not go to infinity). This leads to

$$\tan\delta(L-\beta+\varepsilon)+\tan\delta(L-\beta-\varepsilon)=0 \tag{3.67}$$

and by using the addition formula for the *tangent function*[3] we can see that Equation (3.67) turns out to be

$$\tan 2\delta(L-\beta)\left[1-\tan\delta(L-\beta-\varepsilon)\tan\delta(L-\beta+\varepsilon)\right]=0 \tag{3.68}$$

Then by using Equation (3.67), Equation (3.68) becomes

$$\tan 2\delta(L-\beta)\left[1+\tan^2\delta(L-\beta+\varepsilon)\right]=0 \tag{3.69}$$

and since only the first term can vanish, the critical point can be found by setting $\tan 2\delta(L-\beta)$ equal zero. It is

$$\delta_{critical}=\frac{\pi}{2(L-\beta)} \tag{3.70}$$

This result is depicted in Figure 3-10. It is the result obtained earlier by our prediction process.

It is important to point out that, our search for the critical value of δ as a function of L, β, ε and k^C/k for this problem is based on the condition that F, which is given by Equation (3.55), is not defined. But this cannot hold for the symmetric case (*i.e.*, $\varepsilon=0$), as in this case we found that $T^C_{steady}=E$ and $F=0$. Since $F=0$, then Equation (3.55) cannot be used to find the critical point. Therefore, to solve the case where the central layer is symmetrically placed we must reapply the six boundary conditions given by Equations (3.34) to (3.39) into Equations (3.40), (3.41) and (3.42) taking into account that $F=0$.

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[3] The addition formula is,

$$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$$

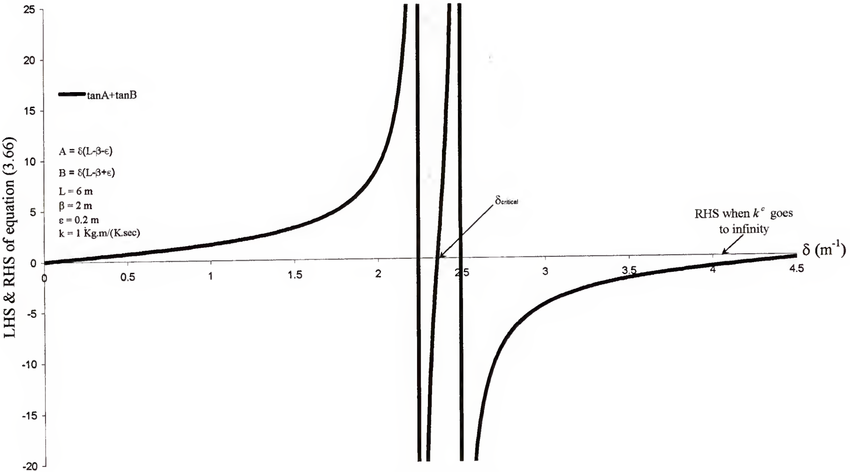

Figure 3-10. Critical value of δ when $k^C \to \infty$.

Thus, we rework the problem using Equations (3.40) to (3.48) where $F = 0$ and $T^C_{steady} = E$ and we get

$$A\cos\delta L - B\sin\delta L - 1 = 0\ ,\ x = -L \tag{3.71}$$

$$A\cos\delta\beta - B\sin\delta\beta - 1 - E = 0\ ,\ x = -\beta \tag{3.72}$$

$$C\cos\delta L + D\sin\delta L - 1 = 0\ ,\ x = L \tag{3.73}$$

$$A\delta\sin\delta\beta + B\delta\cos\delta\beta = 0\ ,\ x = -\beta \tag{3.74}$$

$$C\cos\delta\beta + D\sin\delta\beta - 1 - E = 0\ ,\ x = \beta \tag{3.75}$$

and

$$-C\delta\sin\delta\beta + D\delta\cos\delta\beta = 0\ ,\ x = \beta \tag{3.76}$$

Then, solve Equations (3.71) and (3.74) to find the constants A and B and solve Equations (3.73) and (3.76) to find the constants C and D. From this we find that $A = C$ and $B = -D$. Therefore, Equations (3.72) and (3.75) are *linearly dependent* and the constant E can be found from any one of them.

Then, by substituting the values of A, B, C, D and E into Equations (3.40), (3.41) and (3.42) the base solution can be obtained. It is

$$T_{steady} = -\frac{\cos\delta(x+\beta)}{\cos\delta(L-\beta)} - 1\ ,\ \ -L < x < -\beta \tag{3.77}$$

$$T_{steady} = \frac{\cos\delta(x-\beta)}{\cos\delta(L-\beta)} - 1\ ,\ \beta < x < L \tag{3.78}$$

and

$$T^C_{steady} = \frac{1}{\cos\delta(L-\beta)} - 1\ ,\ \ -\beta < x < \beta \tag{3.79}$$

Based on this the critical value of δ (*i.e.*, the value of δ where the output variable T_{steady} goes to infinity), can be found. This happens when the denominator of the first term on the right-hand side of Equations (3.77), (3.78) and (3.79) vanishes (*i.e.*, when $\cos\delta(L-\beta)=0$), and the critical point is then given as

$$\delta_{critical} = \frac{\pi}{2(L-\beta)} \tag{3.80}$$

This agrees with our earlier prediction.

We can summarize what we have done thus far in the case of the parallel plane wall problem. At first a simple case with no central layer was worked out and plausibility arguments were advanced from which $\delta_{critical}$ could be predicted when changes in geometry and thermal conductivity were made. These predictions were then verified by formal methods. What we learnt is that even for these simple problems the manipulations that determine the critical points are not ordinarily simple. The problem becomes even more complicated if by introducing the displacement of the central region temperature variation in another dimension is created. This is precisely what occurs in the case of the distorted annulus (*i.e.*, a pipe containing a central rod), problem as we shall see in the next chapter.

Now one way used to deal with harder problems is to change the geometry slightly and to use domain perturbation methods. This approach could then be tested on the simple problem given above before it is used on a more complicated problem. However it turns out that, when $\delta = \delta_{critical}$ it is very difficult to apply the perturbation technique directly to the simple equations. To explain the reason behind this let us consider the symmetric problem as our reference state which, as we know, has its own $\delta_{critical}$. Then to find $\delta_{critical}$ for the perturbed problem, where the central layer is slightly offset, we must expand about the solution of the reference state but that might have already blown up. This happens when $\delta_{critical}$ of the perturbed geometry is greater than it is for the reference state and it means that, we are trying to expand around a case that does not exist. On account of this, we cannot see the effect of the perturbation on the thermal stability of the problem. On the other hand, if the solution of the reference state does not blown up (*i.e.*, $\delta_{critical}$ of the perturbed geometry is less than it is of the symmetric one), then the perturbation terms, the correction terms that account for the displacement, in the expansion must blow up[4] and hence the perturbation series makes no sense, as the correction terms must be smaller than the solution of the reference state. Because of this we cannot see the effect of the perturbation on the thermal stability of the problem. Based on these two observations we can see that no conclusion can be drawn from applying the domain perturbation method directly to the equations of the simple problem and that might be due to the fact that the unboundedness or nonexistence of solutions is being used to test for the critical conditions.

Because of the difficulties associated with applying a domain perturbation method directly to the equations themselves in order to find out under what conditions a variable becomes infinitely large, another approach is taken. Note that these problems are *linear* no matter the geometry and if they are treated as *initial value problems* then for δ beyond its critical value an arbitrary initial condition should become unbounded. What this means is that if the problems were solved by eigenfunction expansion an eigenvalue would be positive. The critical value of δ should then correspond to the greatest eigenvalue just becoming zero and no singularity would have to be dealt with. Thus, the way to proceed is to set up the eigenvalue problem and solve it for a simple reference case, say with the layer at the center. The eigenvalue problem with a displaced layer is then solved by perturbing the solution to this problem about the reference state. This provides a way to get a solution as no "infinities" ought to be encountered. The eigenvalue method is illustrated in the following section, however it is not carried out to reproduce all our earlier results. It is applied here to the layer free case and the symmetric case. Its use in non-symmetric cases will be presented in the next chapter when the rod and pipe problem is taken up.

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[4] Remember that the critical value of δ is identified by infinite maximum temperature.

3.3 The Eigenvalue Problem

To get the eigenvalue problem first define $T' = T(x,t) - T_{steady}(x)$. Then substitute $T = T' + T_{steady}$ into the unsteady equations (*i.e.*, Equations (3.28), (3.29) and (3.30)), and make use of the Equations (3.31), (3.32) and (3.33) for T_{steady} to get

$$\frac{1}{\alpha}\frac{\partial T'}{\partial t} = \frac{\partial^2 T'}{\partial x^2} + \delta^2 T' \, , \quad -L < x < -\beta + \varepsilon \tag{3.81}$$

$$\frac{1}{\alpha}\frac{\partial T'}{\partial t} = \frac{\partial^2 T'}{\partial x^2} + \delta^2 T' \, , \quad \beta + \varepsilon < x < L \tag{3.82}$$

and

$$\frac{1}{\alpha^C}\frac{\partial T'^C}{\partial t} = \frac{\partial^2 T'^C}{\partial x^2} \, , \quad -\beta + \varepsilon < x < \beta + \varepsilon \tag{3.83}$$

where at the boundaries and at the surfaces separating the regions we must have

$$T'(x = -L) = 0 \tag{3.84}$$

$$T'(x = -\beta + \varepsilon) = T'^C(x = -\beta + \varepsilon) \tag{3.85}$$

$$T'(x = L) = 0 \tag{3.86}$$

and

$$T'(x = \beta + \varepsilon) = T'^{C}(x = \beta + \varepsilon) \tag{3.87}$$

as well as

$$k\frac{dT'}{dx} = k^{C}\frac{dT'^{C}}{dx} \;,\; x = -\beta + \varepsilon \tag{3.88}$$

and

$$k\frac{dT'}{dx} = k^{C}\frac{dT'^{C}}{dx} \;,\; x = \beta + \varepsilon \tag{3.89}$$

Keep in mind that for simplicity in notation the scaled temperatures T^{*} and T^{*C} and the scaled time t^{*} appearing in Equations (3.28), (3.29) and (3.30) has been replaced by T, T^{C} and t, respectively.

Now to get the eigenvalue problem let us apply the separation of variables method into Equations (3.81), (3.82) and (3.83). But before doing so let us make it clear that T' and T'^{C} are the same variable but it is defined over different regions and that α is function of x, as its value changes from one region to another. To use the separation of variable method, let us define T' and T'^{C} as

$$T' = e^{-\lambda^2 t}\psi \;,\; -L < x < -\beta + \varepsilon$$

$$T' = e^{-\lambda^2 t}\psi \;,\; \beta + \varepsilon < x < L$$

and

$$T'^{C} = e^{-\lambda^2 t}\psi \;,\; -\beta + \varepsilon < x < \beta + \varepsilon$$

where $e^{-\lambda^2 t}$ is the growth rate at which the temperatures in the three regions grow. Then, let us substitute these definitions back into Equations (3.81), (3.82) and (3.83) to obtain

$$\frac{d^2\psi}{dx^2}+\left(\delta^2+\frac{\lambda^2}{\alpha}\right)\psi=0\,,\quad -L<x<-\beta+\varepsilon \tag{3.90}$$

$$\frac{d^2\psi}{dx^2}+\left(\delta^2+\frac{\lambda^2}{\alpha}\right)\psi=0\,,\quad \beta+\varepsilon<x<L \tag{3.91}$$

and

$$\frac{d^2\psi^C}{dx^2}+\frac{\lambda^2}{\alpha^C}\psi^C=0\,,\quad -\beta+\varepsilon<x<\beta+\varepsilon \tag{3.92}$$

where $-\lambda^2$ is the growth rate constant, the eigenvalue, and where ψ is the eigenfunction. The critical value of δ, $\delta_{critical}$, is reached when the growth rate constant, $-\lambda^2$, becoming zero. However, These equations can be written in a simpler form by defining

$$\sigma^2=\left(\delta^2+\frac{\lambda^2}{\alpha}\right) \tag{3.93}$$

and

$$\Lambda^2=\left(\frac{\lambda^2}{\alpha^C}\right) \tag{3.94}$$

Then, the eigenvalue equations become

$$\frac{d^2\psi}{dx^2}+\sigma^2\psi=0\,,\quad -L<x<-\beta+\varepsilon \tag{3.95}$$

$$\frac{d^2\psi}{dx^2}+\sigma^2\psi=0\,,\quad \beta+\varepsilon<x<L \tag{3.96}$$

and

$$\frac{d^2\psi^C}{dx^2}+\Lambda^2\psi^C=0\,,\quad -\beta+\varepsilon<x<\beta+\varepsilon \tag{3.97}$$

where σ^2 is an output that describes the *strength of the thermal diffusion*. It is always positive and it can be redefined as

$$\sigma^2 = \left(\delta^2 + \frac{\lambda^2}{\alpha}\right) = \left(\delta^2 + \frac{\alpha^C}{\alpha}\Lambda^2\right) \tag{3.98}$$

The boundary conditions are

$$\psi(x = -L) = 0 \tag{3.99}$$

$$\psi(x = -\beta + \varepsilon) = \psi^C(x = -\beta + \varepsilon) \tag{3.100}$$

$$\psi(x = L) = 0 \tag{3.101}$$

and

$$\psi(x = \beta + \varepsilon) = \psi^C(x = \beta + \varepsilon) \tag{3.102}$$

as well as

$$k\frac{d\psi}{dx} = k^C\frac{d\psi^C}{dx}\ , \quad x = -\beta + \varepsilon \tag{3.103}$$

and

$$k\frac{d\psi}{dx} = k^C\frac{d\psi^C}{dx}\ , \quad x = \beta + \varepsilon \tag{3.104}$$

Equations (3.95), (3.96) and (3.97) are linear, homogenous, second order ODEs which can be solved to give

$$\psi = A\cos\sigma x + B\sin\sigma x\ , \quad -L < x < -\beta + \varepsilon \tag{3.105}$$

$$\psi = C\cos\sigma x + D\sin\sigma x\ , \quad \beta + \varepsilon < x < L \tag{3.106}$$

and

$$\psi^C = E\cos\Lambda x + F\sin\Lambda x\ , \quad -\beta + \varepsilon < x < \beta + \varepsilon \tag{3.107}$$

Now to find the constants A, B, C, D, E and F substitute these expressions into the six boundary conditions to obtain

$$A\cos\sigma L - B\sin\sigma L = 0 \;,\; x = -L \tag{3.108}$$

$$\begin{aligned} &A\cos\sigma(\beta-\varepsilon) - B\sin\sigma(\beta-\varepsilon) \\ &\qquad - E\cos\Lambda(\beta-\varepsilon) + F\sin\Lambda(\beta-\varepsilon) = 0 \;,\; x = -\beta+\varepsilon \end{aligned} \tag{3.109}$$

$$C\cos\sigma L + D\sin\sigma L = 0 \;,\; x = L \tag{3.110}$$

$$\begin{aligned} &C\cos\sigma(\beta+\varepsilon) + D\sin\sigma(\beta+\varepsilon) \\ &\qquad - E\cos\Lambda(\beta+\varepsilon) - F\sin\Lambda(\beta+\varepsilon) = 0 \;,\; x = \beta+\varepsilon \end{aligned} \tag{3.111}$$

$$\begin{aligned} &A\sigma\sin\sigma(\beta-\varepsilon) + B\sigma\cos\sigma(\beta-\varepsilon) \\ &\qquad - k\left[E\Lambda\sin\Lambda(\beta-\varepsilon) + F\Lambda\cos\Lambda(\beta-\varepsilon)\right] = 0 \;,\; x = -\beta+\varepsilon \end{aligned} \tag{3.112}$$

and

$$\begin{aligned} &-C\sigma\sin\sigma(\beta+\varepsilon) + D\sigma\cos\sigma(\beta+\varepsilon) \\ &\qquad - k\left[-E\Lambda\sin\Lambda(\beta+\varepsilon) + F\Lambda\cos\Lambda(\beta+\varepsilon)\right] = 0 \;,\; x = \beta+\varepsilon \end{aligned} \tag{3.113}$$

Various cases can be worked out and they are illustrated in the following sub-sections.

3.3.1 Case 1: The Planar Problem with No Central Layer

This is the case of no non-heat generating layer and it is the simplest of the cases. The eigenvalue problem for this case can be written as

$$\frac{d^2\psi}{dx^2} + \sigma^2\psi = 0 \;\;,\;\; -L < x < L \tag{3.114}$$

This equation is a linear, homogeneous, second order ODE which can be solved to give

$$\psi = A\cos\sigma x + B\sin\sigma x \;,\;\; -L < x < L \tag{3.115}$$

where ψ must satisfy

$$\psi(x=-L)=0 \tag{3.116}$$

and

$$\psi(x=L)=0 \tag{3.117}$$

whence

$$A\cos\sigma L - B\sin\sigma L = 0\ ,\ x=-L \tag{3.118}$$

and

$$A\cos\sigma L + B\sin\sigma L = 0\ ,\ x=L \tag{3.119}$$

Equations (3.118) and (3.119) can be written in matrix form as

$$\begin{bmatrix} \cos\sigma L & \sin\sigma L \\ \cos\sigma L & -\sin\sigma L \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.120}$$

To find a solution other than $A=0$ and $B=0$, the determinant of the above matrix must vanish, which means

$$-2\cos\sigma L\sin\sigma L = 0 \tag{3.121}$$

In other words we must have[5]

$$-\sin 2\sigma L = 0 \tag{3.122}$$

and hence

$$\sigma = \frac{n\pi}{2L}\ ,\ n=0,1,2,3,... \tag{3.123}$$

If $n=1$ then

$$\sigma\big|_{n=1} = \frac{\pi}{2L} \tag{3.124}$$

and Equation (3.120) turns out to be

---

[5] From the double-angle formula (*i.e.*, $\sin 2A = 2\sin A\cos A$).

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.125}$$

Whereupon A can be anything while B must vanish and the eigenfunction ψ_1 is $\cos L\sigma\big|_{n=1}$. Now as $\sigma^2 = \delta^2 + \left(\lambda^2/\alpha\right)$ we have for the growth constant

$$-\frac{\lambda^2}{\alpha} = -\frac{n^2\pi^2}{4L^2} + \delta^2 \tag{3.126}$$

and our solution takes the form

$$T' = A_1\psi_1 e^{-\lambda_1^2 t} + A_2\psi_2 e^{-\lambda_2^2 t} + A_3\psi_3 e^{-\lambda_3^2 t} + \ldots \tag{3.127}$$

As we stated earlier, the critical value of δ (*i.e.*, $\delta_{critical}$), is reached when the growth constant, $-\lambda^2$, is equal to zero. Thus, from Equation (3.126) and at $n = 1$ we find

$$\delta_{critical} = \frac{\pi}{2L} \tag{3.128}$$

This is the result found from the steady state calculations for the same problem solved at the beginning of this chapter. From Equation (3.126) one can see that the eigenvalue, $-\lambda^2$, depends on δ^2 and Figure 3-11 gives a clear idea on how they are related.

To understand Figure 3-11, let us take an arbitrary disturbance and expand it as the sum of the eigenfunctions $\psi_1, \psi_2, \psi_3, \ldots, \psi_n$ where each of these eigenfunctions is multiplied by the growth rate $e^{-\lambda_n^2 t}$, as given by Equation (3.127). Then, referring to the points marked in Figure 3-11, at point 1 the eigenvalues, $-\lambda^2$, for $\psi_1, \psi_2, \psi_3, \ldots, \psi_n$ are negative. Therefore, at point 1 the whole problem is thermally stable since all the growth rates will die. At point 2 the eigenvalue for the ψ_1 part only is positive then the growth rate for ψ_1 will grow while the growth rates for $\psi_2, \psi_3, \ldots, \psi_n$ will die. Therefore, at point 2 the problem is thermally unstable. However, consider the two points located on the

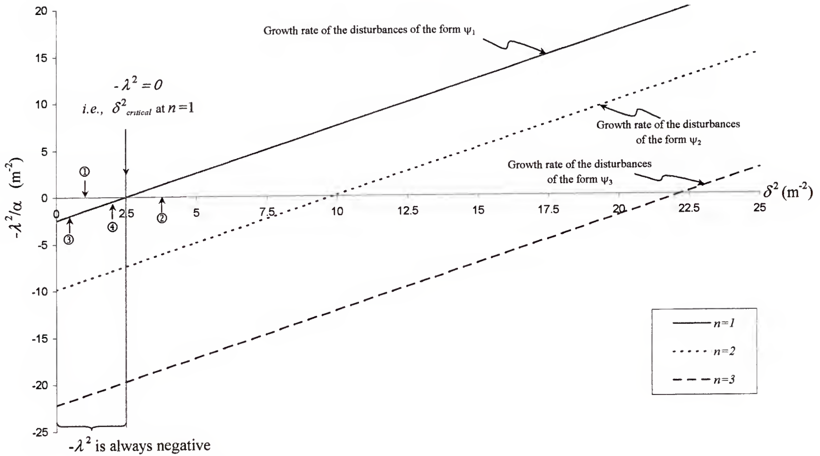

Figure 3-11. Relationship between $-\lambda^2/\alpha$ vs. δ^2 for the planar problem with no-central layer.

$n = 1$ line. On this line we see that, points 3 and 4 are stable points as they are located in the region where the eigenvalues are negative. But point 3 is thermally more stable than point 4 because point 4 is much closer to the critical point, $\delta_{critical}$, than point 3 and its growth rate will die slower.

Based on our understanding of Figure 3-11 and from Equation (3.93) which can be rewritten as

$$-\frac{\lambda^2}{\alpha} = \delta^2 - \sigma^2 \tag{3.129}$$

we can conclude that anything that lowers the growth constant, $-\lambda^2$, makes the system more stable (*i.e.*, allows greater values of δ^2), and since the critical value of δ is at $-\lambda^2 = 0$, then bigger values of δ^2 corresponds to bigger σ^2 which means higher thermal diffusion.

3.3.2 Case 2: The Planar Problem with Central Layer at $\varepsilon = 0$

This is the case when the non heat-generating layer is centered between the parallel plane walls. Now substitute the solution to the domain equations (*i.e.*, Equations (3.105) to (3.107)), into the six boundary conditions taking into consideration that $\varepsilon = 0$ whereupon Equations (3.108) to (3.113) turn out to be

$$A\cos\sigma L - B\sin\sigma L = 0\ ,\ \ x = -L \tag{3.130}$$

$$A\cos\sigma\beta - B\sin\sigma\beta - E\cos\Lambda\beta + F\sin\Lambda\beta = 0\ ,\ \ x = -\beta \tag{3.131}$$

$$C\cos\sigma L + D\sin\sigma L = 0\ ,\ \ x = L \tag{3.132}$$

$$C\cos\sigma\beta + D\sin\sigma\beta - E\cos\Lambda\beta - F\sin\Lambda\beta = 0\ ,\ \ x = \beta \tag{3.133}$$

$$A\sigma\sin\sigma\beta + B\sigma\cos\sigma\beta - k\left[E\Lambda\sin\Lambda\beta + F\Lambda\cos\Lambda\beta\right] = 0\ ,\ \ x = -\beta \tag{3.134}$$

and

$$-C\sigma\sin\sigma\beta + D\sigma\cos\sigma\beta - k\left[-E\Lambda\sin\Lambda\beta + F\Lambda\cos\Lambda\beta\right] = 0\ ,\ x = \beta \qquad (3.135)$$

Then, solve Equations (3.130) and (3.134) to find the constants A and B in terms of E and F and solve Equations (3.132) and (3.135) to find the constants C and D in terms of E and F. The results are

$$A = \frac{k\Lambda\sin\sigma L\left[E\sin\Lambda\beta - F\cos\Lambda\beta\right]}{\sigma\cos\sigma(L-\beta)} \qquad (3.136)$$

$$B = \frac{-k\Lambda\cos\sigma L\left[E\sin\Lambda\beta - F\cos\Lambda\beta\right]}{\sigma\cos\sigma(L-\beta)} \qquad (3.137)$$

$$C = \frac{k\Lambda\sin\sigma L\left[E\sin\Lambda\beta + F\cos\Lambda\beta\right]}{\sigma\cos\sigma(L-\beta)} \qquad (3.138)$$

and

$$D = \frac{k\Lambda\cos\sigma L\left[E\sin\Lambda\beta + F\cos\Lambda\beta\right]}{\sigma\cos\sigma(L-\beta)} \qquad (3.139)$$

Then substitute A and B into Equation (3.131) and C and D into Equation (3.133) to get

$$E\left[\frac{k\Lambda\sin\Lambda\beta\sin\sigma(L-\beta)}{\sigma\cos\sigma(L-\beta)} - \cos\Lambda\beta\right] - F\left[\frac{k\Lambda\cos\Lambda\beta\sin\sigma(L-\beta)}{\sigma\cos\sigma(L-\beta)} - \sin\Lambda\beta\right] = 0 \qquad (3.140)$$

and

$$E\left[\frac{k\Lambda\sin\Lambda\beta\sin\sigma(L-\beta)}{\sigma\cos\sigma(L-\beta)} - \cos\Lambda\beta\right] + F\left[\frac{k\Lambda\cos\Lambda\beta\sin\sigma(L-\beta)}{\sigma\cos\sigma(L-\beta)} + \sin\Lambda\beta\right] = 0 \qquad (3.141)$$

Equations (3.140) and (3.141) can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.142}$$

where

$$a_{11} = \left[\frac{k\Lambda \sin \Lambda\beta \sin \sigma(L-\beta)}{\sigma \cos \sigma(L-\beta)} - \cos \Lambda\beta \right]$$

$$a_{12} = -\left[\frac{k\Lambda \cos \Lambda\beta \sin \sigma(L-\beta)}{\sigma \cos \sigma(L-\beta)} - \sin \Lambda\beta \right]$$

$$a_{21} = \left[\frac{k\Lambda \sin \Lambda\beta \sin \sigma(L-\beta)}{\sigma \cos \sigma(L-\beta)} - \cos \Lambda\beta \right]$$

and

$$a_{22} = \left[\frac{k\Lambda \cos \Lambda\beta \sin \sigma(L-\beta)}{\sigma \cos \sigma(L-\beta)} + \sin \Lambda\beta \right]$$

To find a solution other than $E = 0$ and $F = 0$ the determinant of the matrix given in Equation (3.142) must vanish, which means

$$2\left[\frac{k\Lambda \sin \Lambda\beta \sin \sigma(L-\beta)}{\sigma \cos \sigma(L-\beta)} - \cos \Lambda\beta \right] \left[\frac{k\Lambda \cos \Lambda\beta \sin \sigma(L-\beta)}{\sigma \cos \sigma(L-\beta)} \right] = 0 \tag{3.143}$$

Equation (3.143) can be simplified and written as

$$k\Lambda \tan \Lambda\beta = \sigma \cos \sigma(L-\beta) \tag{3.144}$$

Now on putting $-\lambda^2 = 0$ into Equation (3.144), to get the critical value of δ, we find that

$$\delta_{critical} \cos \delta_{critical}(L-\beta) = 0 \tag{3.145}$$

and hence

$$\delta_{critical} = \frac{n\pi}{2(L-\beta)}, \quad n = 1,3,5... \tag{3.146}$$

Then based on Equation (3.93) which can be written in this case as

$$-\frac{\lambda^2}{\alpha} = -\sigma^2 + \frac{n^2\pi^2}{4(L-\beta)^2} = 0 \tag{3.147}$$

we find

$$\sigma^2 = \frac{n^2\pi^2}{4(L-\beta)^2} \tag{3.148}$$

and the critical value of δ (*i.e.*, $\delta_{critical}$), at $n=1$ will be

$$\delta^2_{critical} = \frac{\pi^2}{4(L-\beta)^2} \tag{3.149}$$

which means

$$\delta_{critical} = \frac{\pi}{2(L-\beta)} \tag{3.150}$$

which is what we expected from our prediction process and from our steady state calculations. From Equation (3.93) which can be written as

$$-\frac{\lambda^2}{\alpha} = -\frac{n^2\pi^2}{4(L-\beta)^2} + \delta^2 \tag{3.151}$$

we can see that $-\lambda^2$ depends on δ^2 and Figure 3-12 gives a clear idea on how they are related. Note that the explanation given earlier on Figure 3-11 can still be used to understand what is going on in Figure 3-12.

Again, from Figure 3-12 and Equation (3.93) we can conclude that as the growth constant, $-\lambda^2$, decreases the system becomes thermally more stable (*i.e.*, allows bigger values of δ^2), and since $\delta_{critical}$ is determined at $-\lambda^2 = 0$ then bigger values of δ^2 corresponds to bigger values of σ^2 which means stronger thermal diffusion.

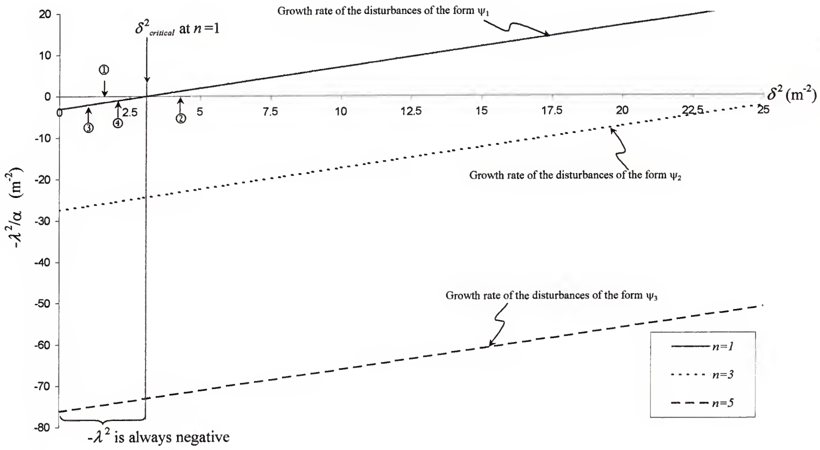

Figure 3-12. Relationship between $-\lambda^2/\alpha$ *vs.* δ^2 for the planar problem with central layer and $\varepsilon = 0$.

Now, from Equations (3.126) and (3.151) we notice that, as n increases the growth rate, $-\lambda^2$, decreases and consequently the thermal stability of the system increases. This can be seen in Figure 3-11 and Figure 3-12. That is the higher n terms are more stable than the lower n terms (*i.e.*, $n=2$ is more stable than $n=1$). This happens because of that, as n increases the strength of the thermal diffusion, σ^2, increases. This can be explained in another way that is the higher the value of n the more wiggly the eigenfunctions will be, because the eigenfunctions are defined in terms of $\cos\sigma x$ and $\sin\sigma x$. And as we know, the wigglier the eigenfunctions are the stronger the diffusion and the stronger the diffusion the more stable the system will be. Therefore, the value of σ^2 can be used as an indication of the stability of the system (*i.e.*, the higher the value of σ^2 is, the more stable the system must be).

From all of the above it is clear that the algebraic manipulations which are involved in trying to get the critical points from the planar geometry either by way of a direct calculation or via the eigenvalue problem solution are not very complicated but that will no longer be the case in a cylindrical geometry. And this task will get even more difficult for the proposed problem of ignition in a distorted annular geometry.

In Appendix B we draw a chart through which one can easily follow up the ways and the methods used in this chapter to solve the planar problem.

From this chapter we can conclude the following:

1. Thin heat-producing regions are thermally more stable than thick heat-producing regions.
2. It is hard to perturb about the solution of a problem which has a critical point defined when $T \to \infty$. Now if perturbations are not necessary as they are not in this chapter then that is not a problem. But if perturbations are required as in harder problems, another method needs to be found, as it is hard to perturb

about infinity. Recall these problems are linear, we can get around this difficulty by going to the eigenvalue problem.

3. For the problems considered in this work, the value of the diffusion strength, σ^2, can be used as an indication on the stability of the system. That is: as the diffusion strength increases the thermal stability of the system increases.

The work so far goes as follows; in chapter 1 the reasons for studying and working on this problem were discussed along with its physics; in chapter 2 we gave an example of how a geometry change could be expected to affect the thermal stability of a heat-producing region and in this chapter the simple planar problem made it clear that the calculations for a more complicated problems must involve a perturbation scheme. In the next chapter the solution to the problem of the thermal stability of a heat-generating annular region will be taken up to see the effect of the central rod being displaced.

CHAPTER 4
THERMAL STABILITY OF A HEAT–GENERATING ANNULAR REGION

This chapter presents the solution of the problem of the thermal stability of a heat-generating annular region, for which the mathematical methods and tools learned from Chapter 2 and Chapter 3 are applied.

This problem is solved under the assumption that heat is being transferred only by conduction; the calculations are done for three different boundary conditions for the inner wall while the boundary condition at the outer wall is held fixed (*i.e.*, it is isothermal). These boundary conditions are:

1. Isothermal inner wall
2. Adiabatic inner wall
3. Perfectly conducting inner wall

This is to show how the thermal stability of this problem can be affected by changing the boundary conditions. The results obtained from the three cases are compared and a conclusion is drawn. The heat transfer assumption is then modified as a little forced convection is added. The reason behind this is to study the effect of each of these two heat-transfer modes (*i.e.*, heat transfer by conduction and by both conduction and convection, on the thermal stability of this problem).

Now by working a simple introductory problem, details can be made clear that otherwise might be obscure in a more complicated problem. This simple problem is to study the thermal stability of a perturbed circular disk (*i.e.*, an ellipse), and its solution is illustrated in the following section.

4.1 Thermal Stability of a Perturbed Circular Disk

Let us begin by describing the physics of this problem. To do this, consider Figure 4-1A. It depicts a heat producing material shaped in the form of a circular disk with a radius R_{out}. Heat is lost to the surroundings through the boundary of the disk which is held at a fixed temperature denoted T_w. The thermal conductivity of the material in the disk is k. The heat source will be taken to depend linearly on the temperature of the heat-generating material.

Now let us perturb this circular disk into the shape of an ellipse holding the area of the disk fixed. Figure 4-1B illustrates the effect of such a perturbation on the shape of the disk. In this figure, a and b are the major axis and the minor axis of the ellipse, respectively, and ε indicates the amount by which the radius of the circle is changed (*i.e.*, $a = R_{out} + \varepsilon$). Then the question arises, does the deformation in the shape of the disk enhance its thermal stability or not? In other words, how high can the strength of the heat source be (*i.e.*, how high can we go with δ^2), without losing the thermal stability of the system? Before the answer to this question is presented let us try to guess it. For a fixed area, the circumference of an ellipse is longer than that of a circle. Based on this we can guess that, the elliptical disk is thermally more stable than the circular disk. To see if this is so the critical value of δ^2 for the two shapes needs to be determined. This will be done in the following section.

The calculation of the critical value of delta. To begin let us consider the perturbed disk illustrated in Figure 4-1B. The temperature rise in this problem can be described by the thermal energy equation

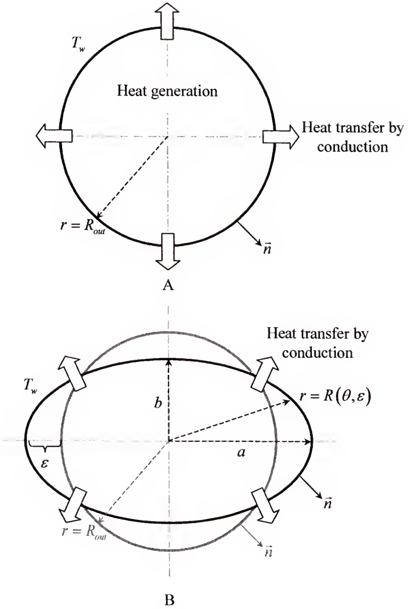

Figure 4-1. Geometry of the auxiliary problem before and after the perturbation. A) The unperturbed geometry (The circular disk). B) The perturbed geometry (The elliptical disk).

$$\rho C_v \frac{\partial T}{\partial t} = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}\right] + Q(T) \tag{4.1}$$

where ρ and C_v denote the constant density and the heat capacity of the heat-producing material, respectively, T is the temperature field and $Q(T)$ denotes the rate at which heat is produced in the material per unit area. The temperature is subject to two boundary conditions viz.,

$$T(r=R)=T_w \tag{4.2}$$

and

$$T(r=0) \text{ is finite} \tag{4.3}$$

Let $Q(T)$ be a linear function of T (*i.e.*, $Q(T)=Q_o+Q_1(T-T_w)$) and set $\dot{T}=T-T_w$. Based on Equation (4.1) $\dot{T}$ must satisfy

$$\rho C_v \frac{\partial \dot{T}}{\partial t}=k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \dot{T}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \dot{T}}{\partial \theta^2}\right]+Q_o(1+S\dot{T}) \tag{4.4}$$

where $S=Q_1/Q_o$ which is the sensitivity of the heat generation to temperature. Then rewrite Equation (4.4) in partly scaled form by defining $\dot{T}=\overline{T}\ T^*$ and $t=\overline{t}\ t^*$, where $\overline{T}=1/S$ and where $\overline{t}=\rho C_v r^2/k$. Then the equation to be solved is

$$\frac{1}{\alpha}\frac{\partial T^*}{\partial t^*}=\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T^*}{\partial r})+\frac{1}{r^2}\frac{\partial^2 T^*}{\partial \theta^2}+\delta^2(1+T^*) \tag{4.5}$$

where

$$\alpha=\frac{k\overline{t}}{\rho C_v} \text{ and } \delta^2=\frac{Q_o S}{k}$$

The value of δ^2 indicates the strength of the heat source and the linear function $(1+T^*)$ indicates how fast it increases as T^* increases.

By dropping the $*$'s Equation (4.5) will become

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r})+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\delta^2(1+T) \tag{4.6}$$

where

$$T(r = R) = 0 \tag{4.7}$$

and

$$T(r = 0) \text{ is finite} \tag{4.8}$$

and the steady-scaled temperature rise must satisfy

$$0 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T_{steady}}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 T_{steady}}{\partial \theta^2} + \delta^2(1 + T_{steady}) \tag{4.9}$$

The input variables of this problem are R_{out}, k, ε and δ^2. The input variables R_{out}, k and ε are held fixed. The only output variable, at steady state, is therefore T_{steady}. It depends on r and θ.

Based on what we learned from Chapter 2 and Chapter 3, this problem can be solved by the domain perturbation method.[1] Since we want to determine the critical value of δ^2 which is defined when $T \to \infty$, the method cannot be applied directly to the domain equations. Instead we must introduce the corresponding eigenvalue problem and solve it for a reference case, the circular disk. The eigenvalue problem for the elliptic disk is then solved by perturbing the equations about the reference state.

Now, to get the eigenvalue problem define $T' = T(r,\theta,t) - T_{steady}(r,\theta)$. Then substitute $T = T' + T_{steady}$ into the unsteady equation (*i.e.*, Equation (4.6)), and make use of the Equation (4.9) for T_{steady} to get

$$\frac{1}{\alpha}\frac{\partial T'}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T'}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 T'}{\partial \theta^2} + \delta^2 T' \tag{4.10}$$

where

---

[1] Appendix A gives a detailed explanation on how to carry out a domain perturbation.

$$T'(r=R)=0 \tag{4.11}$$

and

$$T'(r=0) \text{ is finite} \tag{4.12}$$

To get the eigenvalue problem let us apply the separation of variables method to Equation (4.10). To do this, define T' as

$$T'(r,\theta,t)=\upsilon(r,\theta)\gamma(t)$$

and substitute this into Equation (4.10) to get two differential equations, one in t and one in r and θ. They are

$$\frac{d\gamma(t)}{dt}+\lambda^2\gamma(t)=0 \tag{4.13}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\upsilon(r,\theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\upsilon(r,\theta)}{\partial\theta^2}+\left(\delta^2+\frac{\lambda^2}{\alpha}\right)\upsilon(r,\theta)=0 \tag{4.14}$$

Equation (4.13) can be solved to give

$$\gamma(t)=\dot{C}e^{-\lambda^2 t} \tag{4.15}$$

where $\dot{C}$ is the integration constant, $e^{-\lambda^2 t}$ is the growth rate and $-\lambda^2$ is the separation constant or the growth rate constant. Equation (4.14) is what we are looking for. It is the eigenvalue problem that describes the temperature rise in the ellipse. Let us rewrite this equation in a simpler form by defining

$$\sigma^2=\left(\delta^2+\frac{\lambda^2}{\alpha}\right), \quad -\frac{\lambda^2}{\alpha}=\delta^2-\sigma^2$$

then it becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\upsilon}{\partial r})+\frac{1}{r^2}\frac{\partial^2\upsilon}{\partial\theta^2}+\sigma^2\upsilon=0 \tag{4.16}$$

which is same as

$$\nabla^2 \upsilon + \sigma^2 \upsilon = 0 \tag{4.17}$$

where σ^2 is the eigenvalue and υ is the eigenfunction that is subject to two boundary conditions viz.,

$$\upsilon(r = R) = 0 \tag{4.18}$$

and

$$\upsilon(r = 0) \text{ is finite} \tag{4.19}$$

Based on what we learned in Chapter 3, anything that lowers the growth constant, $-\lambda^2$, makes the system more stable (*i.e.*, allows greater values of δ^2), and what lowers $-\lambda^2$ is σ^2, the eigenvalue or the strength of the diffusion, and since the critical value of δ^2 is reached when $-\lambda^2 = 0$, then larger values of δ^2 corresponds to larger values σ^2 which means higher diffusion rates. Thus, the effect of perturbing the circular disk on its thermal stability can be shown via the change in the value of σ^2.

Now to solve this problem the elliptical disk (*i.e.*, the perturbed shape), needs to be defined. To do this, let its boundary be denoted as

$$r = R(\theta, \varepsilon) \tag{4.20}$$

Now, the area of a circular disk is given by

$$A_{circle} = \pi R_{out}^2 \tag{4.21}$$

while the area of an ellipse is

$$A_{ellipse} = \pi ab \tag{4.22}$$

But, the area of the disk is to be held fixed for all the cases, thus

$$\pi R_{out}^2 = \pi ab \tag{4.23}$$

while

$$a = R_{out} + \varepsilon = R_{out}\left(1 + \frac{\varepsilon}{R_{out}}\right) \tag{4.24}$$

Then by substituting Equation (4.24) into Equation (4.23) we can find b in terms of R_{out} and ε, that is

$$b = \frac{R_{out}^2}{R_{out} + \varepsilon} = \frac{R_{out}}{1 + \frac{\varepsilon}{R_{out}}} \tag{4.25}$$

Therefore, b can be written as

$$b = R_{out}\left(1 - \frac{\varepsilon}{R_{out}} + \frac{\varepsilon^2}{R_{out}^2} - \ldots\right) \tag{4.26}$$

In addition to this, we know that any point on an ellipse must satisfy

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1 \tag{4.27}$$

where X and Y denote the Cartesian coordinates of the points (R, θ) on an ellipse and in this case they are given in terms of R and θ by

$$X = R\cos\theta \tag{4.28}$$

and

$$Y = R\sin\theta \tag{4.29}$$

Then substitute the values of X and Y into Equation (4.27). This leads to

$$\frac{R^2\cos^2\theta}{a^2} + \frac{R^2\sin^2\theta}{b^2} = 1 \tag{4.30}$$

and multiply this equation by a^2b^2 to get

$$R^2\left[b^2\cos^2\theta + a^2\sin^2\theta\right] = a^2b^2 = R_{out}^4 \tag{4.31}$$

Then, substitute the values of a and b given by Equations (4.24) and (4.26), respectively, into Equation (4.31). As a result of this, R, must satisfy

$$R^2\left[\left(R_{out}\left(1-\frac{\varepsilon}{R_{out}}+\frac{\varepsilon^2}{R_{out}^2}-\ldots\right)\right)^2\cos^2\theta+\left(R_{out}\left(1+\frac{\varepsilon}{R_{out}}\right)\right)^2\sin^2\theta\right]=R_{out}^4 \qquad (4.32)$$

which can be expanded and arranged as

$$\left(\frac{R}{R_{out}}\right)^2\left[1+\frac{2\varepsilon}{R_{out}}\left(\sin^2\theta-\cos^2\theta\right)+\frac{\varepsilon^2}{R_{out}^2}\left(1+2\cos^2\theta\right)+\ldots\right]=1 \qquad (4.33)$$

Equation (4.33) defines the perimeter of the perturbed disk as $R=R(\theta,\varepsilon)$.

Now we must solve our eigenvalue problem by perturbing Equation (4.17) about the reference domain (*i.e.*, the circular disk). To do this, let us assume that ε is small and expand R, υ and σ^2 in powers of ε to obtain problems in $\upsilon_o,\upsilon_1,\upsilon_2,\ldots$ on the reference domain by using the domain perturbation method presented in Appendix A. Then the expansion of R can be written in the form

$$R=R_o+\varepsilon R_1+\frac{\varepsilon^2}{2}R_2+\ldots \qquad (4.34)$$

where

$$R_o=R_{out} \qquad (4.35)$$

and R_1 and R_2 can be found by the substituting Equation (4.34) into Equation (4.33), and expanding $(R/R_{out})^2$, which gives

$$\left(1+2R_1\frac{\varepsilon}{R_{out}}+R_1^2\frac{\varepsilon^2}{R_{out}^2}+R_2R_{out}\frac{\varepsilon^2}{R_{out}^2}+\ldots\right)\left[1+\frac{2\varepsilon}{R_{out}}\left(\sin^2\theta-\cos^2\theta\right)+\right.$$
$$\left.\frac{\varepsilon^2}{R_{out}^2}\left(1+2\cos^2\theta\right)+\ldots\right]=1 \qquad (4.36)$$

From this we obtain at order ε^1

$$R_1 = \frac{dR}{d\varepsilon}(\varepsilon = 0) = \cos 2\theta_o \tag{4.37}$$

and at order ε^2

$$R_2 = \frac{d^2R}{d\varepsilon^2}(\varepsilon = 0) = \frac{-\frac{1}{2} + \frac{3}{2}\cos 4\theta_o - \cos 2\theta_o}{R_o} \tag{4.38}$$

The expansion of υ on the domain, by using the rule given in Appendix A, can be written as

$$\upsilon = \upsilon_o + \varepsilon \upsilon_1 + \frac{\varepsilon^2}{2}\upsilon_2 + \ldots \tag{4.39}$$

and the expansion of σ^2 can be written as

$$\sigma^2 = \sigma_o^2 + \varepsilon \sigma_1^2 + \frac{\varepsilon^2}{2}\sigma_2^2 + \ldots \tag{4.40}$$

The problems to be solved are: At order ε^o, which is the reference problem on the circular disk, the domain equation is

$$\nabla^2 \upsilon_o + \sigma_o^2 \upsilon_o = 0 \tag{4.41}$$

where υ_o must satisfy

$$\upsilon_o(r_o = R_{out}) = 0 \tag{4.42}$$

and

$$\upsilon_o(r_o = 0) \text{ is finite} \tag{4.43}$$

For order ε^1, the domain equation is, again on the circular disk,

$$\nabla^2 \upsilon_1 + \sigma_o^2 \upsilon_1 + \sigma_1^2 \upsilon_o = 0 \tag{4.44}$$

where υ_1 must satisfy

$$\upsilon_1(r_o = R_{out}) = -R_1 \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{out}} \tag{4.45}$$

and

$$\upsilon_1(r_o = 0) \text{ is finite} \tag{4.46}$$

For order ε^2, the domain equation is

$$\frac{1}{2}\nabla^2\upsilon_2 + \frac{1}{2}\sigma_o^2\upsilon_2 + \sigma_1^2\upsilon_1 + \frac{1}{2}\sigma_2^2\upsilon_o = 0 \tag{4.47}$$

where υ_2 must satisfy

$$\upsilon_2(r_o = R_{out}) = -2R_1 \left.\frac{d\upsilon_1}{dr_o}\right|_{r_o=R_{out}} - R_1^2 \left.\frac{d^2\upsilon_o}{dr_o^2}\right|_{r_o=R_{out}} - R_2 \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{out}} \tag{4.48}$$

and

$$\upsilon_2(r_o = 0) \text{ is finite} \tag{4.49}$$

To explain where Equations (4.45) and (4.48) come from[2], notice: the expansion of υ at $r = R(\theta,\varepsilon)$, is given by

$$\upsilon(R,\theta) = \upsilon_o + \varepsilon\left(\upsilon_1 + R_1\frac{d\upsilon_o}{dr_o}\right) + \frac{\varepsilon^2}{2}\left(\upsilon_2 + 2R_1\frac{d\upsilon_1}{dr_o} + R_1^2\frac{d^2\upsilon_o}{dr_o^2} + R_2\frac{d\upsilon_o}{dr_o}\right) + \ldots$$

and $\upsilon(R,\theta)$ must vanish for all values of ε.

As we mentioned earlier, the effect of perturbing the circular disk on its thermal stability can be seen from the change in the value of σ^2. Then to study the change in the value of σ^2, we must find the values of σ_o^2, σ_1^2 and σ_2^2. These values can be found from the solutions of Equations (4.41), (4.44) and (4.47), respectively.

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[2] Note that the perturbed boundary conditions are no longer $\upsilon = 0$ because they try to simulate the original condition $\upsilon = 0$ on the ellipse by putting a condition other than $\upsilon = 0$ on the circle.

Let us start with the υ_o equation, which represents the reference state. By solving this equation the value of σ_o^2 can be found. To do so, let us write υ_o as the product of two factors, viz.,

$$\upsilon_o(r_o,\theta_o) = \mathcal{R}(r_o)\mathcal{H}(\theta_o)$$

and then substitute this into Equation (4.41). This will then give two ordinary differential equations one in θ_o and one in r_o. They are

$$\frac{d^2\mathcal{H}}{d\theta_o^2} + M^2\mathcal{H} = 0 \tag{4.50}$$

and

$$\frac{d^2\mathcal{R}}{dr_o^2} + \frac{1}{r_o}\frac{d\mathcal{R}}{dr_o} + \left(\sigma_o^2 - \frac{M^2}{r_o^2}\right)\mathcal{R} = 0 \tag{4.51}$$

where M^2 is the separation constant.

The solution to the first equation is

$$\mathcal{H} = \hat{A}\cos M\theta_o + \hat{B}\sin M\theta_o \tag{4.52}$$

where $\hat{A}$ and $\hat{B}$ are arbitrary constants. For this case the range of θ_o is $0 \le \theta_o \le 2\pi$; and in this situation no boundary conditions are prescribed at θ_o except the requirement that the temperature and hence the eigenfunction should be periodic in θ_o with period 2π. This condition requires M to be an integer, that is $M = 0,1,2,3,\ldots$. This means the solution for Equation (4.52) correspond first to $M = 0$ then to $M = 1$ and so on.

The solution to the second equation is

$$\mathcal{R} = AJ_M(r_o\sigma_o) + BY_M(r_o\sigma_o) \tag{4.53}$$

where $J_M(r_o\sigma_o)$ and $Y_M(r_o\sigma_o)$ are *Bessel functions*[3] of order M of the first and the second kind, respectively. The constants A, B and σ_o remain to be determined. To do this we turn to the boundary conditions for this equation viz.,

$$\mathcal{R}(r_o = R_{out}) = 0 \tag{4.54}$$

while

$$\mathcal{R}(r_o = 0) \text{ must be finite} \tag{4.55}$$

The second boundary condition tells us that B must be zero and, therefore, A can take any value except zero. Then, the solution of Bessel's equation reduces to

$$\mathcal{R} = AJ_M(r_o\sigma_o) \tag{4.56}$$

and the allowable values of σ_o are the roots of $J_M(r_o\sigma_o) = 0$, where $M = 0,1,2,3,...$. To each M there are many values of σ_o. The lowest one is the smallest root corresponding to $M = 0$. This is the value of σ_o of interest it determines the critical value of δ^2.

The corresponding eigenfunction is

$$\upsilon_o = AJ_o\left(r_o\sigma_o\right) \tag{4.57}$$

Two derivatives of this eigenfunction[4] will be needed later. They are

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[3] It is important to mention that, the $Y_p(\xi)$ functions become infinite as ξ becomes zero. More information on Bessel's differential equation and Bessel functions and their properties can be found in McLachlan, N.W., 1955, Bessel Functions for Engineers, Oxford University Press, 2nd Ed., Oxford. and in Watson, G.N., 1995, A Treatise on the Theory of Bessel Functions, Cambridge University Press, 2nd Ed., New York.

[4] Formulas for the derivatives of Bessel functions are (Mickley, H.S., Sherwood, T.K. and Reed, C.E., 1957, Applied Mathematics in Chemical Engineering, McGraw-Hill, 2nd Ed., New York.),

$$\frac{d}{dx}\left[Z_p(\alpha x)\right] = \begin{cases} \alpha Z_{p-1}(\alpha x) - \dfrac{p}{x} Z_p(\alpha x) \\ -\alpha Z_{p+1}(\alpha x) + \dfrac{p}{x} Z_p(\alpha x) \end{cases}, \quad Z = J, Y$$

$$\frac{d\upsilon_o}{dr_o} = A[-\sigma_o J_1(r_o\sigma_o)] \tag{4.58}$$

and

$$\frac{d^2\upsilon_o}{dr_o^2} = -\sigma_o A\left[-\sigma_o J_2(r_o\sigma_o) + \frac{1}{r_o} J_1(r_o\sigma_o)\right] \tag{4.59}$$

The next step is to find the values of σ_1^2 and σ_2^2. By obtaining these values we can determine σ^2 defined earlier as

$$\sigma^2 = \sigma_o^2 + \varepsilon\sigma_1^2 + \frac{\varepsilon^2}{2}\sigma_2^2 + \ldots$$

and draw a conclusion about how the perturbation affects the thermal stability of the system.

Now, to find σ_1^2 we must solve the first order problem

$$\nabla^2\upsilon_1 + \sigma_o^2\upsilon_1 + \sigma_1^2\upsilon_o = 0$$

where υ_1 satisfies

$$\upsilon_1(r_o = R_{out}) = -R_1 \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o = R_{out}}$$

and

$$\upsilon_1(r_o = 0) \text{ is finite}$$

Before solving for υ_1 let us prove that $\sigma_1^2 = 0$. To do this, multiply Equation (4.44) by υ_o, which gives

$$\upsilon_o\nabla^2\upsilon_1 + \sigma_o^2\upsilon_o\upsilon_1 + \sigma_1^2\upsilon_o^2 = 0 \tag{4.60}$$

and multiply Equation (4.41) by υ_1, to get

$$\upsilon_1\nabla^2\upsilon_o + \sigma_o^2\upsilon_o\upsilon_1 = 0 \tag{4.61}$$

Then, subtract Equation (4.61) from Equation (4.60), to obtain

$$\upsilon_o\nabla^2\upsilon_1 - \upsilon_1\nabla^2\upsilon_o + \sigma_1^2\upsilon_o^2 = 0 \tag{4.62}$$

Equation (4.62) can be written in a simpler form using the following relationship

$$A\nabla^2 B = \nabla\cdot\left(A\nabla B\right) - \nabla A\cdot\nabla B$$

to get

$$\nabla\cdot\left(\upsilon_o\nabla\upsilon_1\right) - \nabla\cdot\left(\upsilon_1\nabla\upsilon_o\right) + \sigma_1^2\upsilon_o^2 = 0 \tag{4.63}$$

Now let us integrate Equation (4.63) over the reference domain to get,

$$\iint\limits_{area}\left(\nabla\cdot\left(\upsilon_o\nabla\upsilon_1\right) - \nabla\cdot\left(\upsilon_1\nabla\upsilon_o\right) + \sigma_1^2\upsilon_o^2\right)dA = 0 \tag{4.64}$$

and then by use of the *divergence theorem*[5], Equation (4.64) can be written, first as

$$\int\limits_{perimeter}\left(\vec{n}\cdot\left(\upsilon_o\nabla\upsilon_1\right) - \vec{n}\cdot\left(\upsilon_1\nabla\upsilon_o\right)\right)ds + \iint\limits_{area}\left(\sigma_1^2\upsilon_o^2\right)dA = 0 \tag{4.65}$$

and then, using $ds = d\theta_o R_{out}$ and $dA = d\theta_o r_o dr_o$, as

$$-\int_0^{R_{out}}\int_0^{2\pi}\left(\sigma_1^2\upsilon_o^2\right)d\theta_o r_o dr_o = R_{out}\int_0^{2\pi}\left(\left(\upsilon_o\vec{n}\cdot\nabla\upsilon_1\right) - \left(\upsilon_1\vec{n}\cdot\nabla\upsilon_o\right)\right)d\theta_o \tag{4.66}$$

where $\vec{n}$ is the unit outward normal to the circle viz.,

$$\vec{n} = \vec{i}_r \tag{4.67}$$

Then, because $\upsilon_o = 0$ and $\upsilon_1 = -R_1\left(d\upsilon_o/dr_o\right)$ at $r_o = R_{out}$ Equation (4.66) reduces to

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[5] The divergence theorem can be expressed as

$$\iint\limits_{area}\left(\nabla\cdot A\right)dA = \int\limits_{perimeter}\left(\vec{n}\cdot A\right)ds$$

$$\int_0^{R_{out}}\int_0^{2\pi}\left(\sigma_1^2\upsilon_o^2\right)d\theta_o r_o dr_o = -R_{out}\int_0^{2\pi} R_1\left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{out}} d\theta_o \tag{4.68}$$

Now, $\left(d\upsilon_o/dr_o\right)\Big|_{r_o=R_{out}}$ is independent of θ_o and R_1 is given by $R_1=\cos 2\theta_o$ hence Equation (4.68) becomes

$$\int_0^{R_{out}}\int_0^{2\pi}\left(\sigma_1^2\upsilon_o^2\right)d\theta_o r_o dr_o = -R_{out}\left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{out}}\int_0^{2\pi}\cos 2\theta_o d\theta_o = 0 \tag{4.69}$$

whence

$$\int_0^{R_{out}}\int_0^{2\pi}\left(\sigma_1^2\upsilon_o^2\right)d\theta_o r_o dr_o = 0 \tag{4.70}$$

Because υ_o^2 is not zero, σ_1^2 must be zero. From this we can conclude that σ_1^2 has no effect on σ^2. This is the expected result. Indeed if we resolve this problem by taking a to be the minor axis and writing $a = R_{out}+\varepsilon$ where ε is now *negative* then we obtain again $\sigma^2=\sigma_o^2+\varepsilon\sigma_1^2$ but it would appear that σ^2 is increased in the first case, decreased in the second. But the two problems are the same and to get the same answer σ_1^2 must be zero. Likewise we expect that $\sigma_3^2,\sigma_5^2,\ldots$ to be zero. Hence to learn the effect of a deformation we must go to second order in ε. But we will need υ_1 and to do this we must go on and determine υ_1 by setting $\sigma_1^2=0$ and solving

$$\nabla^2\upsilon_1+\sigma_o^2\upsilon_1=\frac{\partial^2\upsilon_1}{\partial r_o^2}+\frac{1}{r_o}\frac{\partial\upsilon_1}{\partial r_o}+\frac{1}{r_o^2}\frac{\partial^2\upsilon_1}{\partial\theta_o^2}+\sigma_o^2\upsilon_1=0 \tag{4.71}$$

where υ_1 must satisfy

$$\upsilon_1(r_o=R_{out})=-R_1\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{out}}$$

and

$$\upsilon_1(r_o = 0) \text{ is finite}$$

The first thing to observe is that the υ_1 problem is not an eigenvalue problem, since σ_o^2 is known, and it is not homogenous. Its general solution is the sum of a homogeneous solution, $\upsilon_{1_{Homogeneous}}$, where $-R_1\left(d\upsilon_o/dr_o\right)\Big|_{r_o=R_{out}}$ is set to zero and a particular solution, $\upsilon_{1_{Particular}}$. The homogeneous part, which can be obtained by the method used to solve the υ_o problem, is

$$\upsilon_{1_{Homogeneous}} = CJ_o(r_o\sigma_o) = C\upsilon_o \tag{4.72}$$

and because R_1 is a multiple of $\cos 2\theta_o$ we can get a particular solution in the form

$$\upsilon_{1_{Particular}} = f\left(r_o\right)\cos 2\theta_o \tag{4.73}$$

Now, by substituting Equation (4.73) into Equation (4.71) and solving for $f\left(r_o\right)$, a particular solution to the υ_1 problem can be written as

$$\upsilon_{1_{Particular}} = -\frac{J_2(r_o\sigma_o)}{J_2(R_{out}\sigma_o)}\frac{d\upsilon_o}{dr_o}\bigg|_{r_o=R_{out}}\cos 2\theta_o \tag{4.74}$$

Then υ_1 takes the form

$$\upsilon_1 = CJ_o(r_o\sigma_o) - \frac{J_2(r_o\sigma_o)}{J_2(R_{out}\sigma_o)}\frac{d\upsilon_o}{dr_o}\bigg|_{r_o=R_{out}}\cos 2\theta_o \tag{4.75}$$

and we will need, at the next order

$$\frac{d\upsilon_1}{dr_o} = C\frac{dJ_o(r_o\sigma_o)}{dr_o} - \frac{1}{J_2(R_{out}\sigma_o)}\frac{dJ_2(r_o\sigma_o)}{dr_o}\frac{d\upsilon_o}{dr_o}\bigg|_{r_o=R_{out}}\cos 2\theta_o \tag{4.76}$$

Now, to find σ_2^2 we turn to the second order problem which, with $\sigma_1^2 = 0$, is

$$\nabla^2 \upsilon_2 + \sigma_o^2 \upsilon_2 + \sigma_2^2 \upsilon_o = 0 \tag{4.77}$$

where

$$\upsilon_2(r_o = R_{out}) = -2R_1 \left.\frac{d\upsilon_1}{dr_o}\right|_{r_o=R_{out}} - R_1^2 \left.\frac{d^2\upsilon_o}{dr_o^2}\right|_{r_o=R_{out}} - R_2 \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{out}}$$

and

$$\upsilon_2(r_o = 0) \text{ is finite}$$

Now we can find σ_2^2 without solving for υ_2. To do this, first multiply Equation (4.77) by υ_o, which gives

$$\upsilon_o \nabla^2 \upsilon_2 + \sigma_o^2 \upsilon_o \upsilon_2 + \sigma_2^2 \upsilon_o^2 = 0 \tag{4.78}$$

then multiply Equation (4.41) by υ_2, to get

$$\upsilon_2 \nabla^2 \upsilon_o + \sigma_o^2 \upsilon_o \upsilon_2 = 0 \tag{4.79}$$

Now subtract Equation (4.79) from Equation (4.78), to obtain

$$\upsilon_o \nabla^2 \upsilon_2 - \upsilon_2 \nabla^2 \upsilon_o + \sigma_2^2 \upsilon_o^2 = 0 \tag{4.80}$$

or

$$\nabla \cdot (\upsilon_o \nabla \upsilon_2) - \nabla \cdot (\upsilon_2 \nabla \upsilon_o) + \sigma_2^2 \upsilon_o^2 = 0 \tag{4.81}$$

Then integrate this over the domain and use the divergence equation to get

$$-\int_0^{R_{out}} \int_0^{2\pi} (\sigma_2^2 \upsilon_o^2) d\theta_o r_o dr_o = R_{out} \int_0^{2\pi} \left((\upsilon_o \vec{n} \cdot \nabla \upsilon_2) - (\upsilon_2 \vec{n} \cdot \nabla \upsilon_o) \right) d\theta_o \tag{4.82}$$

The first term on the right-hand side of Equation (4.82) vanishes because $\upsilon_o = 0$ at $r_o = R_{out}$ whence using the condition satisfied by υ_2 at $r_o = R_{out}$ we get

$$\int_0^{R_{out}}\int_0^{2\pi}\left(\sigma_2^2\upsilon_o^2\right)d\theta_o r_o dr_o =$$
$$-R_{out}\int_0^{2\pi}\left(2R_1\frac{d\upsilon_1}{dr_o}\frac{d\upsilon_o}{dr_o}+R_1^2\frac{d^2\upsilon_o}{dr_o^2}\frac{d\upsilon_o}{dr_o}+R_2\left(\frac{d\upsilon_o}{dr_o}\right)^2\right)_{r_o=R_{out}}d\theta_o \tag{4.83}$$

and on using

$$\left.\frac{d^2\upsilon_o}{dr_o^2}\right|_{r_o=R_{out}}=-\frac{1}{R_{out}}\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{out}},\quad R_1=\cos 2\theta_o,\ R_2=\frac{-\frac{1}{2}+\frac{3}{2}\cos 4\theta_o-\cos 2\theta_o}{R_o}$$

and

$$\frac{d\upsilon_1}{dr_o}=C\frac{dJ_o(r_o\sigma_o)}{dr_o}-\cos 2\theta_o\frac{1}{J_2(R_{out}\sigma_o)}\frac{dJ_2(r_o\sigma_o)}{dr_o}\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{out}}$$

Equation (4.83) becomes

$$\int_0^{R_{out}}\int_0^{2\pi}\left(\sigma_2^2\upsilon_o^2\right)d\theta_o r_o dr_o=-R_{out}\int_0^{2\pi}\left(2C\cos 2\theta_o\frac{dJ_o\left(r_o\sigma_o\right)}{dr_o}\frac{d\upsilon_o}{dr_o}\right.$$
$$-2\frac{\cos^2 2\theta_o}{J_2\left(R_{out}\sigma_o\right)}\frac{dJ_2\left(r_o\sigma_o\right)}{dr_o}\left(\frac{d\upsilon_o}{dr_o}\right)^2-\frac{\cos^2 2\theta_o}{R_{out}}\left(\frac{d\upsilon_o}{dr_o}\right)^2 \tag{4.84}$$
$$\left.+\frac{-\frac{1}{2}+\frac{3}{2}\cos 4\theta_o-\cos 2\theta_o}{R_{out}}\left(\frac{d\upsilon_o}{dr_o}\right)^2\right)_{r_o=R_{out}}d\theta_o$$

and on carrying out the integration the right-hand side of this equation becomes

$$RHS=2\pi\left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{out}}\left(R_{out}\frac{\left(\sigma_o J_1(R_{out}\sigma_o)-\frac{2}{R_{out}}J_2(R_{out}\sigma_o)\right)}{J_2(R_{out}\sigma_o)}+1\right) \tag{4.85}$$

where the term in C disappear, while on using $J_o\left(R_{out}\sigma_o\right)=0$ the left-hand side[6] turns to be

$$LHS = 2\pi\sigma_2^2 \frac{R_{out}^2}{2\sigma_o^2}\left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{out}} \tag{4.86}$$

whence

$$\sigma_2^2 = \frac{2\sigma_o^2}{R_{out}^2}\left(\frac{R_{out}\left(\sigma_o J_1(R_{out}\sigma_o)-\dfrac{2}{R_{out}}J_2(R_{out}\sigma_o)\right)}{J_2(R_{out}\sigma_o)}+1\right) \tag{4.87}$$

Now, by using the formula[7]

$$Z_p(\alpha x)=\frac{\alpha x}{2p}\left[Z_{p+1}(\alpha x)+Z_{p-1}(\alpha x)\right] \quad , Z=J,Y$$

we can write

$$J_1(R_{out}\sigma_o)=\frac{R_{out}\sigma_o}{2}J_2(R_{out}\sigma_o) \tag{4.88}$$

because $J_o\left(R_{out}\sigma_o\right)=0$, then Equation (4.87) reduces to

$$\sigma_2^2 = \frac{\sigma_o^2}{R_{out}^2}\left(\left(\sigma_o R_{out}\right)^2-2\right) \tag{4.89}$$

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[6] Done by using the following formula (Spiegel, M.R. and Liu, J., 1998, Mathematical Handbook of Formulas and Tables, McGraw-Hill, 2nd Ed., New York.),

$$\int_{\xi_1}^{\xi_2} Z_p^2(\alpha x)x\,dx = \left[\frac{x^2}{2\alpha^2}\left\{\frac{dZ_p}{dx}(\alpha x)\right\}^2+\frac{x^2}{2}\left(1-\frac{p^2}{\alpha^2x^2}\right)\left\{Z_p(\alpha x)\right\}^2\right]_{\xi_1}^{\xi_2} \quad , Z=J,Y$$

[7] From Spiegel, M.R. and Liu, J., 1998, Mathematical Handbook of Formulas and Tables, McGraw-Hill, 2nd Ed., New York.

This equation can be written in fully scaled form by defining $\hat{\sigma}_o = R_{out}\sigma_o$ and $\hat{\sigma}_2 = R_{out}^2\sigma_2$, then it becomes

$$\hat{\sigma}_2^2 = \hat{\sigma}_o^2\left(\hat{\sigma}_o^2 - 2\right) \tag{4.90}$$

The value of $\hat{\sigma}_o$, the smallest root of $J_o(\hat{\sigma}_o) = 0$, is roughly equal to 2.4, hence the sign of $\hat{\sigma}_2^2$ remains always positive. Therefore, σ_2^2 is always positive.

Now σ^2, the smallest eigenvalue for the perturbed disk, is given in terms of σ_o^2, the smallest eigenvalue for the disk it self, as

$$\sigma^2 = \sigma_o^2 + \frac{\varepsilon^2}{2}\sigma_2^2 + \ldots \tag{4.91}$$

The critical value of δ^2 is always the smallest value of σ^2, hence $\delta_{critical}^2$ for a perturbed disk is always greater than $\delta_{critical}^2$ for a circular disk. Based on this we can conclude that, an elliptical disk is thermally more stable than a circular disk of the same area. This set the background for the problems to come.

4.2 Thermal Stability of a Heat-Generating Annular Region

This section presents a study of heat generation in the region between a rod and a pipe (*i.e.*, in an annular region). We are interesting in the effect of moving the rod off-center. In fact heat exchangers are often build in concentric cylinder form and the question that might arises is: do we need to be careful to get the alignment right? To answer this question the critical value of δ^2 for the on-center case and for the off-center case must be determined. That is our aim in this chapter. We will organize our work along the lines of Chapter 2, Chapter 3 and the previous section, section 4.1.

Now, let us begin by describing the physics of this problem. Consider Figure 4-2. It depicts a heat-producing material confined in the space between two concentric cylinders. The surface temperature of the pipe is T_w. The conditions at the surface of the rod remain to be specified. Inside the rod no heat is being generated. The thermal conductivity of the material of the rod is denoted k^C, a variable whose value can range from zero to infinity. The thermal conductivity of the material between the rod and the pipe is denoted k, and ε denotes the distance by which the rod is shifted from the center of the geometry. The heat source will be taken to depend linearly on the temperature of the heat-generating material. It will be of interest to study the effect on the thermal stability of this geometry as the rod is shifted and its thermal conductivity changes. In other words, we are interested in knowing, for different boundary conditions at the inner wall, if the thermal stability is advanced as the rod is moved off-center.

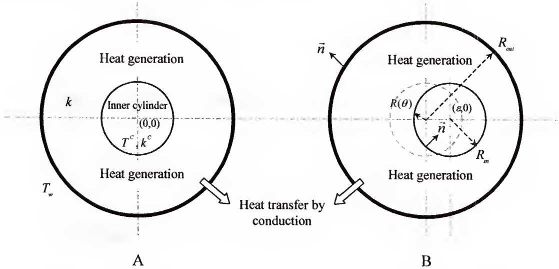

Figure 4-2. Displaced solid rod inside a pipe. A) On-center. B) Off-center.

4.2.1 The Mathematical Problem of Heat Transfer by Conduction

Now, as we learned in Chapter 2, where the rod and the pipe were at different temperatures, a shift in the rod causes the heat-transfer rate by conduction to increase.

But now the shifted rod also creates both a thick and a thin heat-producing region. It is the thick region that would appear to determine the fate of the thermal stability and if the azimuthal temperature gradients from this region cannot conduct heat fast enough then thermal instability will set in earlier in the off-center case. Only a calculation can tell whether the rate of heat production in the distorted annular compartment will be higher than the enhanced heat-transfer rate or the other way around. This problem will be solved for three different boundary conditions at the inner wall.

Now, let us begin with the right part of Figure 4-2. For this geometry the temperature rise with respect to the pipe wall, referring to Figure 4-2B for a definition of the variables, can be described in partly scaled form by

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r})+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\delta^2(1+T)\ ,\ \begin{cases} R(\theta,\varepsilon)<r<R_{out} \\ 0<\theta<2\pi \end{cases} \tag{4.92}$$

Our aim is to find $\delta^2_{critical}$ where $\delta^2_{critical}$ corresponds to an unbounded temperature. When the rod is on-center we can do this by solving the steady form of Equation (4.92) but this would not lead us to a result for the off-center rod by direct domain perturbation. Hence we turn to the eigenvalue problem and deduce $\delta^2_{critical}$ as the value of δ^2 when $-\lambda^2=0$.

Then, by following the steps given in section 4.1, the eigenvalue problem is found to be

$$\nabla^2\upsilon+\sigma^2\upsilon=0 \tag{4.93}$$

where again $\sigma^2=\delta^2+\lambda^2/\alpha$ and where for an isothermal outer wall υ must satisfy

$$\upsilon(r=R_{out})=0 \tag{4.94}$$

Now we are going to consider a variety of conditions at the inner wall (*i.e.*, isothermal), adiabatic and perfectly conducting rod. In these cases υ must satisfy

$$\upsilon\left(r=R(\theta,\varepsilon)\right)=0 \tag{4.95}$$

or

$$\vec{n}\cdot\nabla\upsilon\left(r=R(\theta,\varepsilon)\right)=0 \tag{4.96}$$

or

$$\upsilon\left(r=R(\theta,\varepsilon)\right)=\upsilon^{C} \tag{4.97}$$

where υ^{C} remains to be defined. In no case do we need a heat conduction equation for the rod.

For this problem (*i.e.*, Figure 4-2B), the outer wall (*i.e.*, the pipe wall), presents no problem; it is the inner wall (*i.e.*, the rod wall), whose surface is displaced. For such a case, the surface of the rod is given by $r=R(\theta,\varepsilon)$ where R is expanded as[8]

$$R=R_o+\varepsilon R_1+\frac{\varepsilon^2}{2}R_2+\ldots \tag{4.98}$$

where

$$R_o=R_{in},\ R_1=\cos\theta_o \text{ and } R_2=-\frac{\sin^2\theta_o}{R_o}$$

Again, the solution of the reference case, the on-center case, will be denoted by affixing a subscript zero to its variables while the subscripts one and two will denote terms in the expansion to first and second order.

Let $\vec{n}$ denote the outward normal to the rod. Then $\vec{n}$ is given by

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[8] For details on the expansions of R and $\vec{n}$ see Chapter 2.

$$\vec{n} = \frac{\vec{i}_r - \vec{i}_\theta \dfrac{R'}{R}}{\sqrt{1 + \left(\dfrac{R'}{R}\right)^2}} \tag{4.99}$$

and it can be expanded as

$$\vec{n} = \vec{n}_o + \varepsilon \vec{n}_1 + \frac{\varepsilon^2}{2} \vec{n}_2 + \ldots \tag{4.100}$$

where, using the expansion of R, see Appendix D, we obtain

$$\vec{n}_o = \vec{i}_r , \quad \vec{n}_1 = \vec{i}_\theta \frac{\sin\theta_o}{R_{in}} \text{ and } \vec{n}_2 = -\vec{i}_r \frac{\sin^2\theta_o}{2R_{in}^2}$$

The expansion of $\vec{n}$ is needed in the expansion of $\vec{n}.\nabla\upsilon$. The expansion of $\nabla\upsilon$ at the surface of the rod is

$$\begin{aligned} \nabla\upsilon(R,\theta) = \nabla\upsilon_o + \varepsilon\left(\nabla\upsilon_1 + R_1 \frac{d\nabla\upsilon_o}{dr_o}\right) \\ + \frac{\varepsilon^2}{2}\left(\nabla\upsilon_2 + 2R_1 \frac{d\nabla\upsilon_1}{dr_o} + R_1^2 \frac{d^2\nabla\upsilon_o}{dr_o^2} + R_2 \frac{d\nabla\upsilon_o}{dr_o}\right) + \ldots \end{aligned} \tag{4.101}$$

Hence the expansion of $\vec{n}.\nabla\upsilon$ at $r = R(\theta,\varepsilon)$ is given by

$$\begin{aligned} \vec{n}.\nabla\upsilon = -\vec{i}_r \cdot \nabla\upsilon_o - \vec{i}_r \cdot \varepsilon\left(\nabla\upsilon_1 + R_1 \frac{d\nabla\upsilon_o}{dr_o}\right) \\ -\vec{i}_r \cdot \frac{\varepsilon^2}{2}\left(\nabla\upsilon_2 + 2R_1 \frac{d\nabla\upsilon_1}{dr_o} + R_1^2 \frac{d^2\nabla\upsilon_o}{dr_o^2} + R_2 \frac{d\nabla\upsilon_o}{dr_o}\right) \\ - \varepsilon\vec{i}_\theta \frac{\sin\theta_o}{R_{in}} \cdot \nabla\upsilon_o - \varepsilon\vec{i}_\theta \frac{\sin\theta_o}{R_{in}} \cdot \varepsilon\left(\nabla\upsilon_1 + R_1 \frac{d\nabla\upsilon_o}{dr_o}\right) + \frac{\varepsilon^2}{2} \vec{i}_r \frac{\sin^2\theta_o}{R_{in}^2} \cdot \nabla\upsilon_o - \ldots \end{aligned} \tag{4.102}$$

which reduces to

$$\vec{n}.\nabla \upsilon = -\frac{\partial \upsilon_o}{\partial r_o} - \varepsilon\left(\frac{\partial \upsilon_1}{\partial r_o} + \cos\theta_o \frac{\partial^2 \upsilon_o}{\partial r_o^2} + \sin\theta_o \frac{\partial \upsilon_o}{\partial \theta_o} \right)$$
$$-\frac{\varepsilon^2}{2}\left(\frac{\partial \upsilon_2}{\partial r_o} + 2\cos\theta_o \frac{\partial^2 \upsilon_1}{\partial r_o^2} + \cos^2\theta_o \frac{\partial^3 \upsilon_o}{\partial r_o^3} - \frac{\sin^2\theta_o}{R_{in}} \frac{\partial^2 \upsilon_o}{\partial r_o^2} \right. \tag{4.103}$$
$$\left. +2\frac{\sin\theta_o}{R_{in}^2}\frac{\partial \upsilon_1}{\partial \theta_o} + 2\frac{\sin\theta_o \cos\theta_o}{R_{in}^2}\frac{\partial^2 \upsilon_o}{\partial \theta_o \partial r_o} - \frac{\sin^2\theta_o}{2R_{in}^2}\frac{\partial \upsilon_o}{\partial r_o} \right) - \dots$$

Therefore, in the next three sections where heat is lost by conduction the perturbed problems are obtained at various orders. At order ε^o, the domain equation is

$$\nabla^2 \upsilon_o + \sigma_o^2 \upsilon_o = 0 \tag{4.104}$$

where

$$\upsilon_o(r_o = R_{out}) = 0 \tag{4.105}$$

For order ε^1 we have

$$\nabla^2 \upsilon_1 + \sigma_o^2 \upsilon_1 + \sigma_1^2 \upsilon_o = 0 \tag{4.106}$$

where

$$\upsilon_1(r_o = R_{out}) = 0 \tag{4.107}$$

For order ε^2 we have

$$\frac{1}{2}\nabla^2 \upsilon_2 + \frac{1}{2}\sigma_o^2 \upsilon_2 + \sigma_1^2 \upsilon_1 + \frac{1}{2}\sigma_2^2 \upsilon_o = 0 \tag{4.108}$$

where

$$\upsilon_2(r_o = R_{out}) = 0 \tag{4.109}$$

In all these problems σ_1^2 will be zero and in the first and the second cases the proof is just like the perturbed disk problem and it will not be given.

We are now ready to solve some problems. Our first one would be the case where the rod wall is at a fixed temperature and it is presented in the following.

4.2.1.1 Isothermal inner wall

In this case, the case of constant temperature walls, the heat generated in the annular region is removed by conduction to the walls, which serve as heat sinks. As depicted in Figure 4-3A, this case has two heat sinks, an inner sink and an outer sink, and, therefore, the maximum temperature is expected to be located somewhere away from the walls[9]. When the inner rod is perturbed, as depicted in Figure 4-3A, there will be two factors by which the thermal stability of this problem can be affected. They are the wider region to the left of the rod, which reduces the thermal stability as it produces more heat, and the θ-diffusion, which enhances the thermal stability because it can carry heat from the thick to the thin region and, therefore, to the nearby sinks.

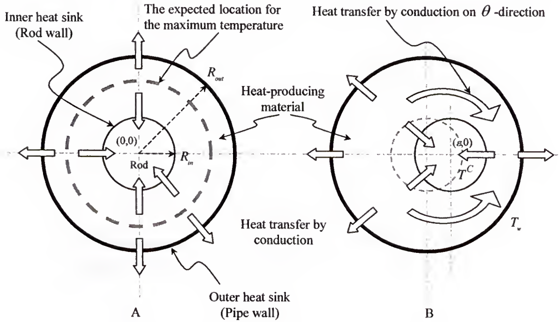

Figure 4-3. Heat transfer between a rod and a pipe both at fixed temperatures. A) On-center. B) Off-center.

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[9] In this case whether the maximum temperature is closer to the inner wall or outer wall cannot be guessed and the reason for this is discussed in Appendix C.

Now for this case the boundary condition[10] at the inner wall is given by $\upsilon = 0$ at $r = R(\theta, \varepsilon)$. Hence we have at order ε^o

$$\upsilon_o(r_o = R_{in}) = 0 \tag{4.110}$$

At order ε^1 we have

$$\upsilon_1(r_o = R_{in}) = -R_1 \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o = R_{in}} \tag{4.111}$$

At order ε^2 we have

$$\upsilon_2(r_o = R_{in}) = -2R_1 \left.\frac{d\upsilon_1}{dr_o}\right|_{r_o = R_{in}} - R_1^2 \left.\frac{d^2\upsilon_o}{dr_o^2}\right|_{r_o = R_{in}} - R_2 \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o = R_{in}} \tag{4.112}$$

At order ε^o the eigenfunctions are products of

$$\hat{A}\cos M\theta_o + \hat{B}\sin M\theta_o$$

and

$$AJ_M(r_o\sigma_o) + BY_M(r_o\sigma_o)$$

where $M = 0, 1, 2, \ldots$. The eigenvalues are roots of

$$J_M(R_{out}\sigma_o)Y_M(R_{in}\sigma_o) - J_M(R_{in}\sigma_o)Y_M(R_{out}\sigma_o) = 0 \tag{4.113}$$

The most dangerous perturbation corresponds to the eigenfunction having the least eigenvalue, this is the smallest root of the above equation when $M = 0$. It depends on R_{out}/R_{in}. Then, for $M = 0$, the foregoing equation can be scaled by defining $\hat{\sigma}_o = R_{in}\sigma_o$ and written as

$$J_o\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right)Y_o(\hat{\sigma}_o) - J_o(\hat{\sigma}_o)Y_o\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right) = 0 \tag{4.114}$$

---

[10] Note that the perturbed boundary conditions are no longer isothermal because they try to simulate isothermal by putting a condition other than isothermal on the reference domain.

and the smallest values of $\hat{\sigma}_o$'s at different values of R_{out}/R_{in} are presented in Table 4-1.

Table 4-1. Values of $\hat{\sigma}_o$ at various R_{out}/R_{in} for isothermal case.

R_{out}/R_{in}	$\hat{\sigma}_o$
5	0.763
10	0.331
15	0.210

The eigenfunction is found as

$$\upsilon_o = C\left[-Y_o(R_{out}\sigma_o)J_o(r_o\sigma_o) + Y_o(r_o\sigma_o)J_o(R_{out}\sigma_o)\right] \tag{4.115}$$

where C is a constant that can have any value except zero. This equation can be written in fully scaled form, that is

$$\upsilon_o = C\left[-Y_o(\frac{R_{out}}{R_{in}}\hat{\sigma}_o)J_o(\hat{r}_o\hat{\sigma}_o) + Y_o(\hat{r}_o\hat{\sigma}_o)J_o(\frac{R_{out}}{R_{in}}\hat{\sigma}_o)\right] \tag{4.116}$$

where $\hat{r}_o = r_o/R_{in}$ and therefore

$$\frac{d\upsilon_o}{d\hat{r}_o} = C\left[-Y_o(\frac{R_{out}}{R_{in}}\hat{\sigma}_o)\frac{dJ_o(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o} + \frac{dY_o(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o}J_o(\frac{R_{out}}{R_{in}}\hat{\sigma}_o)\right] \tag{4.117}$$

This eigenfunction will simplify our calculation because at $M = 0$ there is only one eigenfunction for each eigenvalue but when M is not equal to zero there are two eigenfunctions for each eigenvalue and when we perturb the domain the case $M = 0$ will still give one eigenfunction for each eigenvalue while the other case, M not equal to zero, will give two eigenvalues each with one eigenfunction and this would make our calculations more complicated.

Again the critical value of δ^2 corresponds to $-\lambda^2 = 0$ and hence $\hat{\delta}^2_{critical}$, where $\hat{\delta} = R_{in}\delta$, is given by $\hat{\sigma}^2$ and one can note from Table 4-1 that thick heat-producing regions are thermally less stable than thin heat-producing regions. This result is expected as it agrees with what we learned from the planar problem discussed in Chapter 3.

Now we have the eigenfunctions in the reference state and the corresponding eigenvalues. These must be corrected as the rod is moved off-center and so we turn to the calculation of σ_1^2 and σ_2^2 which will allows us to estimate the effect of the displacement on σ^2 via

$$\sigma^2 = \sigma_o^2 + \varepsilon\sigma_1^2 + \frac{\varepsilon^2}{2}\sigma_2^2 + \ldots$$

Now, $\sigma_1^2 = 0$. This result is due to the symmetry of the problem. Because it should make no difference in the answer whether the rod is shifted to the right (*i.e.*, $\varepsilon > 0$), or to the left (*i.e.*, $\varepsilon < 0$). Hence to learn the effect of a displacement we must go to second order of ε. But there we will need υ_1 which can be determined using

$$\nabla^2\upsilon_1 + \sigma_o^2\upsilon_1 = 0 \tag{4.118}$$

where υ_1 must satisfy

$$\upsilon_1(r_o = R_{out}) = 0$$

and

$$\upsilon_1(r_o = R_{in}) = -R_1 \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o = R_{in}}$$

Notice that the υ_1 problem is not an eigenvalue problem since σ_o^2 is known and its general solution is the sum of a homogeneous solution where $-R_1\left(d\upsilon_o/dr_o\right)\big|_{r_o=R_{in}}$ is set to zero and a particular solution. The homogeneous part is just

$$\upsilon_{1_{Homogeneous}} = \dot{C}\left[-Y_o(R_{out}\sigma_o)J_o(r_o\sigma_o) + Y_o(r_o\sigma_o)J_o(R_{out}\sigma_o)\right] = \dot{C}\upsilon_o \tag{4.119}$$

and, because R_1 is a multiple of $\cos\theta_o$, we can get a particular solution in the form

$$\upsilon_{1_{Particular}} = f\left(r_o\right)\cos\theta_o \tag{4.120}$$

Now, by substituting Equation (4.120) into Equation (4.118) and solving for $f\left(r_o\right)$, a particular solution to the υ_1 problem can be written as

$$\upsilon_{1_{Particular}} = \cos\theta_o\left[C_1 J_1(r_o\sigma_o) + C_2 Y_1(r_o\sigma_o)\right] \tag{4.121}$$

where the constants C_1 and C_2 can be determined by using the boundary conditions. Doing this we have

$$\upsilon_1 = \dot{C}\upsilon_o + \cos\theta_o\left[\frac{Y_1(R_{out}\sigma_o)J_1(r_o\sigma_o) - Y_1(r_o\sigma_o)J_1(R_{out}\sigma_o)}{-Y_1(R_{out}\sigma_o)J_1(R_{in}\sigma_o) + Y_1(R_{in}\sigma_o)J_1(R_{out}\sigma_o)}\right]\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{in}} \tag{4.122}$$

and $d\upsilon_1/dr_o$, needed later, is given by

$$\frac{d\upsilon_1}{dr_o} = \dot{C}\frac{d\upsilon_o}{dr_o} + \cos\theta_o\omega_I\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{in}} \tag{4.123}$$

where

$$\omega_I = \left[\frac{Y_1(R_{out}\sigma_o)\dfrac{dJ_1(r_o\sigma_o)}{dr_o} - \dfrac{dY_1(r_o\sigma_o)}{dr_o}J_1(R_{out}\sigma_o)}{-Y_1(R_{out}\sigma_o)J_1(R_{in}\sigma_o) + Y_1(R_{in}\sigma_o)J_1(R_{out}\sigma_o)}\right]$$

and where the subscript I stands for the word isothermal. Notice that the homogeneous part of the υ_1 will have no effect on the value of σ_2^2 because it will enter the calculation multiplied by $\cos\theta_o$ which when integrated over $0, 2\pi$ will vanishes. So, we drop this term henceforth.

At the second order of ε the problem is

$$\nabla^2\upsilon_2 + \sigma_o^2\upsilon_2 + \sigma_2^2\upsilon_o = 0 \tag{4.124}$$

where

$$\upsilon_2(r_o = R_{out}) = 0$$

and

$$\upsilon_2(r_o = R_{in}) = -2R_1 \left.\frac{d\upsilon_1}{dr_o}\right|_{r_o=R_{in}} - R_1^2 \left.\frac{d^2\upsilon_o}{dr_o^2}\right|_{r_o=R_{in}} - R_2 \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{in}}$$

Now we can get σ_2^2 without determining υ_2. To do this multiply Equation (4.124) by υ_o, multiply Equation (4.104) by υ_2 and subtract the second from the first to obtain

$$\nabla \cdot (\upsilon_o \nabla \upsilon_2) - \nabla \cdot (\upsilon_2 \nabla \upsilon_o) + \sigma_2^2 \upsilon_o^2 = 0 \tag{4.125}$$

Then, integrate Equation (4.125) over the reference domain and then use the divergence theorem, to get

$$\int\limits_{perimeter} \left(\vec{n} \cdot (\upsilon_o \nabla \upsilon_2) - \vec{n} \cdot (\upsilon_2 \nabla \upsilon_o)\right) ds + \iint\limits_{area} \left(\sigma_2^2 \upsilon_o^2\right) dA = 0 \tag{4.126}$$

Because the perimeter consist of two circles, an inner circle at $r_o = R_{in}$ with $ds = d\theta_o R_{in}$ and outer circle at $r_o = R_{out}$ with $ds = d\theta_o R_{out}$, and because $dA = d\theta_o r_o dr_o$, Equation (4.126) can be arranged as

$$\begin{aligned} -\int_{R_{in}}^{R_{out}} \int_0^{2\pi} \left(\sigma_2^2 \upsilon_o^2\right) d\theta_o r_o dr_o &= \int_0^{2\pi} R_{out} \left(\vec{n} \cdot (\upsilon_o \nabla \upsilon_2) - \vec{n} \cdot (\upsilon_2 \nabla \upsilon_o)\right) d\theta_o \bigg|_{r_o=R_{out}} \\ &+ \int_0^{2\pi} R_{in} \left(\vec{n} \cdot (\upsilon_o \nabla \upsilon_2) - \vec{n} \cdot (\upsilon_2 \nabla \upsilon_o)\right) d\theta_o \bigg|_{r_o=R_{in}} \end{aligned} \tag{4.127}$$

Now υ_o and υ_2 vanish at $r_o = R_{out}$ and υ_o vanishes at $r_o = R_{in}$, hence only the last term on the right-hand side survives, whence

$$\int_{R_{in}}^{R_{out}}\int_0^{2\pi}\left(\sigma_2^2\upsilon_o^2\right)d\theta_o r_o dr_o =$$

$$R_{in}\int_0^{2\pi}\left(2R_1\frac{d\upsilon_1}{dr_o}\frac{d\upsilon_o}{dr_o}+R_1^2\frac{d^2\upsilon_o}{dr_o^2}\frac{d\upsilon_o}{dr_o}+R_2\left(\frac{d\upsilon_o}{dr_o}\right)^2\right)_{r_o=R_{in}}d\theta_o \tag{4.128}$$

where R_1 and R_2 are given by

$$R_1=\cos\theta_o,\quad R_2=-\frac{\sin^2\theta_o}{R_o}$$

Now, to simplify this equation, we use

$$\left.\frac{d^2\upsilon_o}{dr_o^2}\right|_{r_o=R_{in}}=-\frac{1}{R_{in}}\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{in}} \tag{4.129}$$

and Equation (4.123), at $r_o=R_{in}$, without the homogeneous part, viz.,

$$\left.\frac{d\upsilon_1}{dr_o}\right|_{r_o=R_{in}}=\cos\theta_o\,\omega_I\big|_{r_o=R_{in}}\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{in}}$$

to get

$$\sigma_2^2\int_{R_{in}}^{R_{out}}\int_0^{2\pi}\left(\upsilon_o^2\right)d\theta_o r_o dr_o=\left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{in}}\int_0^{2\pi}\left(2R_{in}\,\omega_I\big|_{r_o=R_{in}}\cos^2\theta_o-1\right)d\theta_o \tag{4.130}$$

This is our equation for σ_2^2. It has been obtained without solving for υ_2. On evaluating the integral of the left-hand side[11] we obtain

$$\int_{R_{in}}^{R_{out}}\int_0^{2\pi}\upsilon_o^2 d\theta_o r_o dr_o=\pi\left[\frac{R_{out}^2}{\sigma_o^2}\left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{out}}-\frac{R_{in}^2}{\sigma_o^2}\left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{in}}\right] \tag{4.131}$$

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[11] Done by using the following formula (Spiegel, M.R. and Liu, J., 1998, Mathematical Handbook of Formulas and Tables, McGraw-Hill, 2nd Ed., New York.),

$$\int_{\xi_1}^{\xi_2}Z_p^2(\alpha x)x dx=\left[\frac{x^2}{2\alpha^2}\left\{\frac{dZ_p}{dx}(\alpha x)\right\}^2+\frac{x^2}{2}\left(1-\frac{p^2}{\alpha^2x^2}\right)\left\{Z_p(\alpha x)\right\}^2\right]_{\xi_1}^{\xi_2}\quad ,\ Z=J,Y$$

which is positive and, on evaluating the integral of the right-hand side, we obtain

$$\sigma_2^2 = \frac{2(\omega_I|_{r_o=R_{in}} R_{in} - 1)\left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{in}}}{\left[\frac{R_{out}^2}{\sigma_o^2}\left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{out}} - \frac{R_{in}^2}{\sigma_o^2}\left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{in}}\right]} \tag{4.132}$$

In scaled form this is

$$\hat{\sigma}_2^2 = \frac{2(\hat{\omega}_I|_{\hat{r}_o=1} - 1)\left(\frac{d\upsilon_o}{d\hat{r}_o}\right)^2_{\hat{r}_o=1}}{\left[\left(\frac{R_{out}}{R_{in}}\right)^2 \frac{1}{\hat{\sigma}_o^2}\left(\frac{d\upsilon_o}{d\hat{r}_o}\right)^2_{\hat{r}_o=\frac{R_{out}}{R_{in}}} - \frac{1}{\hat{\sigma}_o^2}\left(\frac{d\upsilon_o}{d\hat{r}_o}\right)^2_{\hat{r}_o=1}\right]} \tag{4.133}$$

where $\hat{\sigma}_2^2 = R_{in}^4 \sigma_2^2$ and where

$$\frac{d\upsilon_o}{d\hat{r}_o} = C\left[-Y_o\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right)\frac{dJ_o(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o} + \frac{dY_o(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o} J_o\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right)\right]$$

and

$$\hat{\omega}_I = \left[\frac{Y_1\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right)\frac{dJ_1(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o} - \frac{dY_1(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o} J_1\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right)}{-Y_1\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right)J_1(\hat{\sigma}_o) + Y_1(\hat{\sigma}_o)J_1\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right)}\right]$$

Notice that $\hat{\sigma}_2^2$, like $\hat{\sigma}_o^2$, depends only on R_{out}/R_{in}.

Now the sign of $\hat{\sigma}_2^2$ depends only on the sign of the factor $(\hat{\omega}_I|_{\hat{r}_o=1} - 1)$ because the signs of all the other components of Equation (4.133) are known, they are positive. Unfortunately, we have been unable to determine the sign of $(\hat{\omega}_I|_{\hat{r}_o=1} - 1)$ without a calculation. But the calculations show that it is always negative, hence $\hat{\sigma}_2^2$ must be negative.

Because $\hat{\delta}^2_{critical} = \hat{\sigma}^2$ and because

$$\hat{\sigma}^2 = \hat{\sigma}_o^2 + \frac{\varepsilon^2}{2}\hat{\sigma}_2^2 + \dots \tag{4.134}$$

tells us what happens on displacing the rod. The displacement decreases the value of $\hat{\sigma}^2$, and hence the system becomes less stable. It appears that the θ-diffusion cannot offset the effect of the wider region to the left of the shifted rod. Based on this we can conclude that, at constant walls temperatures, a symmetric annular region is thermally more stable than a distorted one. Now we turn to the adiabatic rod.

4.2.1.2 Adiabatic inner wall

In this case, the surface of the rod wall is now adiabatic (*i.e.*, $k^C = 0$), and the surface of the pipe is still isothermal. The heat generated in the annular region is removed by conduction to the outer wall which serves as the only heat sink, as depicted in Figure 4-4A, and, therefore, the maximum temperature is expected to move inward and located near the rod wall. Again, when the inner rod is perturbed, as depicted in Figure 4-4B, the thermal stability of this problem can be affected by two factors. They are the wider region to the left of the rod and the θ-diffusion. Notice that in this case the larger region might now become more important because the maximum temperature moves inward while the θ-diffusion can now carry heat only to the pipe wall.

Because the surface of the rod wall is an insulator, the surface variable $\vec{n}$ comes into play and it changes as the rod is displaced. Hence it needs to be expanded in powers of ε to turn the adiabatic condition on the current domain into the corresponding condition on the reference domain (the symmetric annular region). The expansion of $\vec{n}$, the unit outward normal to the annular region, in powers of ε is given in Appendix D.

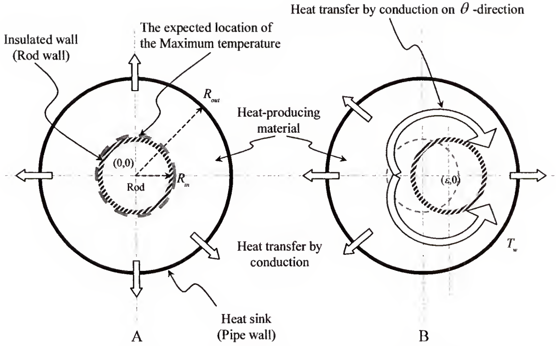

Figure 4-4. Heat transfer between an adiabatic rod and a pipe at fixed temperature. A) On-center. B) Off-center.

Now for this case the boundary condition[12] at the inner wall is given by $\vec{n}\cdot\nabla\upsilon = 0$ at $r = R(\theta,\varepsilon)$. Hence we have at order ε^o

$$\frac{d\upsilon_o}{dr_o}(r_o = R_{in}) = 0 \tag{4.135}$$

At order ε^1

$$\frac{\partial\upsilon_1}{\partial r_o}(r_o = R_{in}) = -R_1\left.\frac{\partial^2\upsilon_o}{\partial r_o^2}\right|_{r_o=R_{in}} - \sin\theta_o\left.\frac{\partial\upsilon_o}{\partial\theta_o}\right|_{r_o=R_{in}} \tag{4.136}$$

At order ε^2

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[12] Note that the perturbed boundary conditions are no longer adiabatic because they try to simulate adiabatic by putting a condition other than adiabatic on the reference domain.

$$\frac{\partial \upsilon_2}{\partial r_o}(r_o = R_{in}) = -2R_1 \left.\frac{\partial^2 \upsilon_1}{\partial r_o^2}\right|_{r_o=R_{in}} - R_1^2 \left.\frac{\partial^3 \upsilon_o}{\partial r_o^3}\right|_{r_o=R_{in}} - R_2 \left.\frac{\partial^2 \upsilon_o}{\partial r_o^2}\right|_{r_o=R_{in}} - 2\frac{\sin\theta}{R_{in}^2}\left.\frac{\partial \upsilon_1}{\partial \theta}\right|_{r_o=R_{in}} \tag{4.137}$$

At order ε^o the base eigenvalues σ_o^2 are determined. Then, picking out one of these, its corrections because of displacement of the rod are determined. The first order problem produces σ_1^2 and the second order problem produces σ_2^2. The plan is the same as earlier, only the boundary condition at the rod is different.

At order ε^o the eigenfunctions are products of

$$\hat{A}\cos M\theta_o + \hat{B}\sin M\theta_o$$

and

$$AJ_M(r_o\sigma_o) + BY_M(r_o\sigma_o)$$

where $M = 0,1,2,\ldots$. The eigenvalues are roots of

$$J_M(R_{out}\sigma_o)\left.\frac{dY_M(r_o\sigma_o)}{dr_o}\right|_{r_o=R_{in}} - \left.\frac{dJ_M(r_o\sigma_o)}{dr_o}\right|_{r_o=R_{in}} Y_M(R_{out}\sigma_o) = 0 \tag{4.138}$$

Again, the most dangerous perturbation corresponds to the eigenfunction having the least eigenvalue (the smallest root of the above equation when $M = 0$). It depends on R_{out}/R_{in}. Then, for $M = 0$, the foregoing equation can be scaled and written as

$$J_o\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right)\left.\frac{dY_o(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o}\right|_{\hat{r}_o=1} - \left.\frac{dJ_o(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o}\right|_{\hat{r}_o=1} Y_o\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right) = 0 \tag{4.139}$$

and the smallest values of $\hat{\sigma}_o$'s at different values of R_{out}/R_{in} are presented in Table 4-2.

Table 4-2. Values of $\hat{\sigma}_o$ at various R_{out}/R_{in} for adiabatic case.

R_{out}/R_{in}	$\hat{\sigma}_o = R_{in}\sigma_o$
5	0.514
10	0.245
15	0.162

The eigenfunction is found as

$$\upsilon_o = C\left[-Y_o(R_{out}\sigma_o)J_o(r_o\sigma_o) + Y_o(r_o\sigma_o)J_o(R_{out}\sigma_o)\right] \tag{4.140}$$

where C is an undetermined constant. In fully scaled form this equation becomes

$$\upsilon_o = C\left[-Y_o(\frac{R_{out}}{R_{in}}\hat{\sigma}_o)J_o(\hat{r}_o\hat{\sigma}_o) + Y_o(\hat{r}_o\hat{\sigma}_o)J_o(\frac{R_{out}}{R_{in}}\hat{\sigma}_o)\right] \tag{4.141}$$

and therefore

$$\frac{d\upsilon_o}{d\hat{r}_o} = C\left[-Y_o(\frac{R_{out}}{R_{in}}\hat{\sigma}_o)\frac{dJ_o(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o} + \frac{dY_o(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o}J_o(\frac{R_{out}}{R_{in}}\hat{\sigma}_o)\right] \tag{4.142}$$

Again, $\hat{\delta}^2_{critical}$ corresponds to $-\lambda^2 = 0$ and hence $\hat{\delta}^2_{critical}$ is given by $\hat{\sigma}^2$ and one can notice from Table 4-2 that thick heat-producing regions are thermally less stable than thin heat-producing regions.

Now to correct the values of σ_o^2 for the displacement of the rod we pick one out, here the most dangerous one at $M = 0$, and find σ_1^2 and σ_2^2 so that

$$\sigma^2 = \sigma_o^2 + \varepsilon\sigma_1^2 + \frac{\varepsilon^2}{2}\sigma_2^2 + \ldots$$

Now, $\sigma_1^2 = 0$. This result is due to the symmetry of the problem. Hence to learn the effect of a displacement we must go to second order of ε. But there we will need υ_1 which can be determined by using

$$\nabla^2\upsilon_1 + \sigma_o^2\upsilon_1 = 0 \tag{4.143}$$

where

$$\upsilon_1(r_o = R_{out}) = 0 \tag{4.144}$$

and

$$\frac{d\upsilon_1}{dr_o}(r_o = R_{in}) = -R_1 \left.\frac{d^2\upsilon_o}{dr_o^2}\right|_{r_o=R_{in}} \tag{4.145}$$

Notice that in $\left(d\upsilon_1/dr_o\right)$ at $r_o = R_{in}$, which is defined earlier by Equation (4.136), the term $\sin\theta_o\left(\partial\upsilon_o/\partial\theta_o\right)$ vanishes and this is because υ_o at $M = 0$ is independent on θ_o. Also, notice that the υ_1 problem is not an eigenvalue problem since σ_o^2 is known and its general solution is the sum of a homogeneous solution where $-R_1\left(d^2\upsilon_o/dr_o^2\right)\Big|_{r_o=R_{in}}$ is set to zero and a particular solution. The homogeneous part is

$$\upsilon_{1_{Homogeneous}} = \dot{C}\left[-Y_o(R_{out}\sigma_o)J_o(r_o\sigma_o) + Y_o(r_o\sigma_o)J_o(R_{out}\sigma_o)\right] = \dot{C}\upsilon_o \tag{4.146}$$

and, because R_1 is a multiple of $\cos\theta_o$, we can get a particular solution in the form

$$\upsilon_{1_{Particular}} = f\left(r_o\right)\cos\theta_o \tag{4.147}$$

Now, by substituting Equation (4.147) into Equation (4.143) and solving for $f\left(r_o\right)$, a particular solution to the υ_1 problem can be written as

$$\upsilon_{1_{Particular}} = \cos\theta_o\left[C_1J_1(r_o\sigma_o) + C_2Y_1(r_o\sigma_o)\right] \tag{4.148}$$

where the constants C_1 and C_2 can be determined by using Equations (4.144) and (4.145) . Doing this we have

$$\upsilon_1 = \dot{C}\upsilon_o + \cos\theta_o\omega_A \left.\frac{d^2\upsilon_o}{dr_o^2}\right|_{r_o=R_{in}} \tag{4.149}$$

where

$$\omega_A = \left[\frac{Y_1(R_{out}\sigma_o)J_1(r_o\sigma_o) - Y_1(r_o\sigma_o)J_1(R_{out}\sigma_o)}{-Y_1(R_{out}\sigma_o)\left.\dfrac{dJ_1(r_o\sigma_o)}{dr_o}\right|_{r_o=R_{in}} + \left.\dfrac{dY_1(r_o\sigma_o)}{dr_o}\right|_{r_o=R_{in}} J_1(R_{out}\sigma_o)} \right]$$

and where the subscript A stands for the word adiabatic. As we explained earlier, the homogeneous part of the υ_1 will have no effect on the value of σ_2^2 because it will enter the calculation multiplied by $\cos\theta_o$ which when integrated over $0, 2\pi$ will vanish. So, we drop this term from now on.

At the order ε^o the problem is

$$\nabla^2\upsilon_2 + \sigma_o^2\upsilon_2 + \sigma_2^2\upsilon_o = 0 \tag{4.150}$$

where

$$\upsilon_2(r_o = R_{out}) = 0$$

and

$$\frac{\partial\upsilon_2}{\partial r_o}(r_o = R_{in}) = -2R_1\left.\frac{\partial^2\upsilon_1}{\partial r_o^2}\right|_{r_o=R_{in}} - R_1^2\left.\frac{\partial^3\upsilon_o}{\partial r_o^3}\right|_{r_o=R_{in}} - R_2\left.\frac{\partial^2\upsilon_o}{\partial r_o^2}\right|_{r_o=R_{in}} - 2\left.\frac{\sin\theta}{R_{in}^2}\frac{\partial\upsilon_1}{\partial\theta}\right|_{r_o=R_{in}}$$

Now we can find σ_2^2 without also find υ_2. To do this, multiply Equation (4.150) by υ_o, multiply Equation (4.104) by υ_2 and subtract the second from the first to obtain

$$\nabla\cdot\left(\upsilon_o\nabla\upsilon_2\right) - \nabla\cdot\left(\upsilon_2\nabla\upsilon_o\right) + \sigma_2^2\upsilon_o^2 = 0 \tag{4.151}$$

Then, integrate Equation (4.151) over the reference domain and make use of the divergence theorem to get

$$
\begin{aligned}
-\int_{R_{in}}^{R_{out}}\int_{0}^{2\pi}\left(\sigma_2^2\upsilon_o^2\right)d\theta_o r_o dr_o &= \int_0^{2\pi} R_{out}\left(\vec{n}\cdot\left(\upsilon_o\nabla\upsilon_2\right)-\vec{n}\cdot\left(\upsilon_2\nabla\upsilon_o\right)\right)d\theta_o\Big|_{r_o=R_{out}} \\
&\quad + \int_0^{2\pi} R_{in}\left(\vec{n}\cdot\left(\upsilon_o\nabla\upsilon_2\right)-\vec{n}\cdot\left(\upsilon_2\nabla\upsilon_o\right)\right)d\theta_o\Big|_{r_o=R_{in}}
\end{aligned}
\tag{4.152}
$$

Now υ_o and υ_2 vanish along the pipe wall while $\vec{n}.\nabla\upsilon_o = -\left(d\upsilon_o/dr_o\right)\big|_{r_o=R_{in}}$ is zero along the rod wall. This eliminates three terms on the right-hand side of this equation including the terms in which υ_2 appears. This is important because we do not know υ_2 at $r_o = R_{in}$. Then since $\left(d\upsilon_2/dr_o\right)$ is specified on terms of υ_o and υ_1 on the inner wall we get the equation for σ_2^2, namely

$$
\begin{aligned}
&\sigma_2^2\int_{R_{in}}^{R_{out}}\int_0^{2\pi}\left(\upsilon_o^2\right)d\theta_o r_o dr_o = \\
&R_{in}\,\upsilon_o\big|_{r_o=R_{in}}\int_0^{2\pi}\left(-2R_1\frac{\partial^2\upsilon_1}{\partial r_o^2}-R_1^2\frac{\partial^3\upsilon_o}{\partial r_o^3}-R_2\frac{\partial^2\upsilon_o}{\partial r_o^2}-2\frac{\sin\theta}{R_{in}^2}\frac{\partial\upsilon_1}{\partial\theta}\right)_{r_o=R_{in}}d\theta_o
\end{aligned}
\tag{4.153}
$$

To simplify this result we use the fact that υ_o satisfies

$$
\frac{d^2\upsilon_o}{dr_o^2}+\frac{1}{r_o}\frac{d\upsilon_o}{dr_o}+\sigma_o^2\upsilon_o = 0 \tag{4.154}
$$

and hence also

$$
\frac{d^3\upsilon_o}{dr_o^3}+\frac{1}{r_o}\frac{d^2\upsilon_o}{dr_o^2}-\frac{1}{r_o^2}\frac{d\upsilon_o}{dr_o}+\sigma_o^2\frac{d\upsilon_o}{dr_o} = 0 \tag{4.155}
$$

Now, at $r_o = R_{in}$, υ_o must also satisfy $\left(d\upsilon_o/dr_o\right)=0$ whereupon we have

$$
\frac{d^2\upsilon_o}{dr_o^2}\bigg|_{r_o=R_{in}} = -\sigma_o^2\,\upsilon_o\big|_{r_o=R_{in}} \tag{4.156}
$$

and

$$\left.\frac{d^3\upsilon_o}{dr_o^3}\right|_{r_o=R_{in}} = -\frac{1}{r_o}\left.\frac{d^2\upsilon_o}{dr_o^2}\right|_{r_o=R_{in}} = \frac{1}{R_{in}}\sigma_o^2\,\upsilon_o\big|_{r_o=R_{in}} \tag{4.157}$$

Turning to υ_1, on the right-hand side of Equation (4.153) we have

$$\left.\frac{\partial\upsilon_1}{\partial r_o}\right|_{r_o=R_{in}} = R_1\sigma_o^2\,\upsilon_o\big|_{r_o=R_{in}} \tag{4.158}$$

and we can make use of the fact that υ_1 satisfies

$$\frac{\partial^2\upsilon_1}{\partial r_o^2}+\frac{1}{r_o}\frac{\partial\upsilon_1}{\partial r_o}+\frac{1}{r_o^2}\frac{\partial^2\upsilon_1}{\partial\theta_o^2}+\sigma_o^2\upsilon_1=0 \tag{4.159}$$

to get $\partial^2\upsilon_1/\partial r_o^2$ at $r_o = R_{in}$ via

$$\left.\frac{\partial^2\upsilon_1}{\partial r_o^2}\right|_{r_o=R_{in}} = -\frac{1}{R_{in}}R_1\sigma_o^2\,\upsilon_o\big|_{r_o=R_{in}} - \frac{1}{R_{in}^2}\left.\frac{\partial^2\upsilon_1}{\partial\theta_o^2}\right|_{r_o=R_{in}} - \sigma_o^2\,\upsilon_1\big|_{r_o=R_{in}} \tag{4.160}$$

Hence, we need $\left(\partial\upsilon_1/\partial\theta_o\right)$ in Equation (4.153) and $\left(\partial^2\upsilon_1/\partial\theta_o^2\right)$ in Equation (4.160) both evaluated at $r_o = R_{in}$. But υ_1 at $r_o = R_{in}$ is given by

$$\upsilon_1 = \cos\theta_o\,\omega_A\big|_{r_o=R_{in}}\left.\frac{d^2\upsilon_o}{dr_o^2}\right|_{r_o=R_{in}} = -\omega_A\big|_{r_o=R_{in}}\,\sigma_o^2\,\upsilon_o\big|_{r_o=R_{in}}\cos\theta_o \tag{4.161}$$

and by using this we find

$$\left.\frac{\partial\upsilon_1}{\partial\theta_o}\right|_{r_o=R_{in}} = \omega_A\big|_{r_o=R_{in}}\,\sigma_o^2\,\upsilon_o\big|_{r_o=R_{in}}\sin\theta_o \tag{4.162}$$

and

$$\left.\frac{\partial^2\upsilon_1}{\partial\theta_o^2}\right|_{r_o=R_{in}} = \omega_A\big|_{r_o=R_{in}}\,\sigma_o^2\,\upsilon_o\big|_{r_o=R_{in}}\cos\theta_o \tag{4.163}$$

and hence

$$\left.\frac{d^2\upsilon_1}{dr_o^2}\right|_{r_o=R_{in}} = -\frac{1}{R_{in}}\sigma_o^2\,\upsilon_o\big|_{r_o=R_{in}}\cos\theta_o - \frac{1}{R_{in}^2}\,\omega_A\big|_{r_o=R_{in}}\,\sigma_o^2\,\upsilon_o\big|_{r_o=R_{in}}\cos\theta_o$$
$$+\,\omega_A\big|_{r_o=R_{in}}\,\sigma_o^4\,\upsilon_o\big|_{r_o=R_{in}}\cos\theta_o \qquad (4.164)$$

Now all the θ_o dependence in our equation for σ_2^2 is explicit and we can carry out the integration over θ_o to get

$$\sigma_2^2\int_{R_{in}}^{R_{out}}\upsilon_o^2 r_o\,dr_o = -R_{in}\,\upsilon_o^2\big|_{r_o=R_{in}}\,\sigma_o^4\,\omega_A\big|_{r_o=R_{in}} \qquad (4.165)$$

This is our equation for σ_2^2 and it has been obtained without solving for υ_2. On evaluating the integral of the left-hand side we obtain

$$\int_{R_{in}}^{R_{out}}\upsilon_o^2 r_o\,dr_o = \left[\frac{R_{out}^2}{2\sigma_o^2}\left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{out}} - \frac{R_{in}^2}{2}\left(\upsilon_o\right)^2_{r_o=R_{in}}\right] \qquad (4.166)$$

which is positive. Then σ_2^2 can be written as

$$\sigma_2^2 = \frac{-2R_{in}\,\upsilon_o^2\big|_{r_o=R_{in}}\,\sigma_o^4\,\omega_A\big|_{r_o=R_{in}}}{\left[\dfrac{R_{out}^2}{\sigma_o^2}\left(\dfrac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{out}} - R_{in}^2\left(\upsilon_o\right)^2_{r_o=R_{in}}\right]} \qquad (4.167)$$

and in fully scaled form it becomes

$$\hat{\sigma}_2^2 = \frac{-2\,\upsilon_o^2\big|_{\hat{r}_o=1}\,\hat{\sigma}_o^4\,\hat{\omega}_A\big|_{\hat{r}_o=1}}{\left[\left(\dfrac{R_{out}}{R_{in}}\right)^2\dfrac{1}{\hat{\sigma}_o^2}\left(\dfrac{d\upsilon_o}{d\hat{r}_o}\right)^2_{\hat{r}_o=\frac{R_{out}}{R_{in}}} - \left(\upsilon_o\right)^2_{\hat{r}_o=1}\right]} \qquad (4.168)$$

where

$$\frac{d\upsilon_o}{d\hat{r}_o} = C\left[-Y_o\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right)\frac{dJ_o(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o} + \frac{dY_o(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o}J_o\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right)\right]$$

and

$$\hat{\omega}_A = \left[\frac{Y_1(\frac{R_{out}}{R_{in}}\hat{\sigma}_o)J_1(\hat{r}_o\hat{\sigma}_o) - Y_1(\hat{r}_o\hat{\sigma}_o)J_1(\frac{R_{out}}{R_{in}}\hat{\sigma}_o)}{-Y_1(\frac{R_{out}}{R_{in}}\hat{\sigma}_o)\left.\frac{dJ_1(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o}\right|_{\hat{r}_o=1} + \left.\frac{dY_1(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o}\right|_{\hat{r}_o=1} J_1(\frac{R_{out}}{R_{in}}\hat{\sigma}_o)} \right]$$

Notice that $\hat{\sigma}_2^2$, like $\hat{\sigma}_o^2$, depends only on R_{out}/R_{in}.

Now the sign of $\hat{\sigma}_2^2$ depends only on the sign of $\hat{\omega}_A\big|_{\hat{r}_o=1}$ because the signs of all other components of Equation (4.133) are known, they are positive. Unfortunately, we have been unable to determine the sign of $\hat{\omega}_A\big|_{\hat{r}_o=1}$ without a calculation. But the calculations show it must be positive, hence $\hat{\sigma}_2^2$ must be negative.

Again, because $\hat{\delta}^2_{critical} = \hat{\sigma}^2$ and because

$$\hat{\sigma}^2 = \hat{\sigma}_o^2 + \frac{\varepsilon^2}{2}\hat{\sigma}_2^2 + \ldots \tag{4.169}$$

tells us what happens on displacing the rod. The displacement decreases the value of $\hat{\sigma}^2$, and hence the system becomes less stable. It appears that the θ-diffusion cannot offset the effect of the wider region to the left of the shifted rod. Based on this we can conclude that, at adiabatic inner wall, a symmetric annular region is thermally more stable than a distorted one. Therefore, the perturbation in this case decreases the thermal stability of the annular region.

To study the difference between the previous problem, the constant temperatures walls, and this problem, the adiabatic inner wall, a comparison between the results obtained from each of these two problems is required. This comparison will be made and presented at the end of this section along with the results obtained from the next problem (*i.e.*, the perfectly conducting inner wall). Now let us turn to the perfectly conducting rod.

4.2.1.3 Perfectly conducting rod

In this case, the rod is now perfectly conducting (*i.e.*, $k^C \to \infty$). The heat generated in the annular region is removed by conduction to the outer wall, which serves as the only heat sink. On the on-centered case, as depicted in Figure 4-5A, the maximum temperature will be at the inner wall just as in the adiabatic case, hence it has the same $\delta_{critical}$ as the adiabatic case; smaller than the isothermal case. We saw this earlier in the planar problem. In the off-centered case, depicted in Figure 4-5B, the thermal stability of this problem can be affected by three factors. They are the wider region to the left of the rod, the θ-conduction and the heat conduction through the rod. The third factor can enhance the thermal stability as it can carry heat directly to the thin region. It is only via calculation we can see the roles of these factors and learn how perturbing the perfectly conducting rod can affect the thermal stability of this problem.

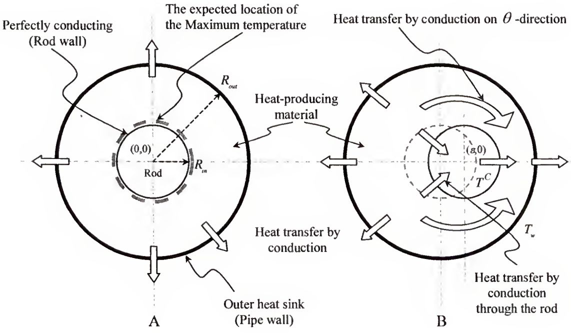

Figure 4-5. Heat transfer between a perfectly conducting rod and a pipe at fixed temperature. A) On-center. B) Off-center.

Because $k^C \to \infty$, the boundary condition at the rod wall needs to be explained. Once that is done the eigenvalue problem on the annular domain can be solved to obtain the values of σ_o^2 and σ^2. This eigenvalue problem is a little different from those in sections 4.2.1.1 and 4.2.1.2 and so some explanation by way of simpler problem is in order. This is given in Appendix E. In this simpler problem we study heat loss from a well-mixed tank by conduction through an attached rod and try to see the roles of the tank and the rod in determining the rate at which temperature perturbation dies out.

Let us write the boundary condition that the temperature would have to satisfy. It is

$$\int\limits_{perimeter} \vec{n} \cdot \vec{q} ds = \pi R_{in}^2 \rho^C C_v^C \frac{dT^C}{dt}, \quad \text{at } r = R_{in} \tag{4.170}$$

where T^C denotes the uniform temperature of the rod, which is equal to T at $r = R_{in}$, $\vec{q} = -k\nabla T$, ρ^C denotes the density of the material of the rod, C_v^C denotes its heat capacity and $\vec{n}$ is the unit outward normal to the surface of the rod.

Now expressing the integral on the left-hand side of Equation (4.170) presents a difficulty when the rod is off-center. It is not easy to decide the element of arc length, ds, on the surface of the rod in terms of coordinate from the center of the pipe. This illustrated in Figure 4-6. In fact even the range of θ depends on ε. To retain the same physics but in a little simpler form let us center our coordinates system on the center of the rod whereupon it appears as though the pipe has been displaced, as depicted in Figure 4-7.

Now, in scaled form Equation (4.170) is

$$\int_0^{2\pi} \left.\frac{\partial T}{\partial r}\right|_{R_{in}} d\theta = \left.\frac{\pi \rho^C C_v^C \alpha}{R_{in} \rho C_v \alpha} \frac{dT^C}{dt}\right|_{R_{in}} \tag{4.171}$$

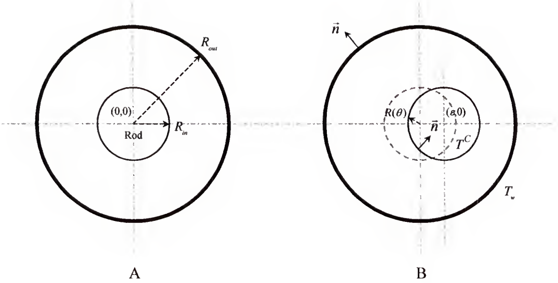

Figure 4-6. Displaced rod with a fixed pipe. A) On-center. B) Off-center.

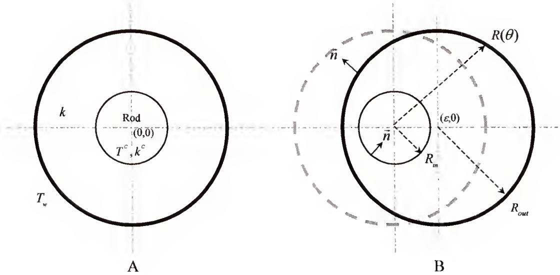

Figure 4-7. Displaced pipe with a fixed rod. A) On-center. B) Off-center.

and hence writing $T = \upsilon e^{-\lambda^2 t}$ we have

$$\int_0^{2\pi} \left.\frac{\partial \upsilon}{\partial r}\right|_{R_{in}} d\theta = \left(-\frac{\lambda^2}{\alpha}\right)\eta \upsilon\big|_{R_{in}} \tag{4.172}$$

where

$$\eta = \frac{\pi \rho^C C_v^C \alpha}{R_{in} \rho C_v}$$

and this is the boundary condition at the rod wall, now not displaced.

The eigenvalue problem then is

$$\nabla^2 \upsilon + \sigma^2 \upsilon = \nabla^2 \upsilon + \left(\delta^2 + \frac{\lambda^2}{\alpha} \right) \upsilon = 0$$

where now

$$\upsilon\left(r = R(\theta, \varepsilon)\right) = 0 \text{ , on the pipe wall} \qquad (4.173)$$

and

$$\int_0^{2\pi} \left. \frac{\partial \upsilon}{\partial r} \right|_{R_{in}} d\theta = \left(-\frac{\lambda^2}{\alpha} \right) \eta \left. \upsilon \right|_{R_{in}} \text{, on the rod wall} \qquad (4.174)$$

Notice that υ at $r = R_{in}$ is independent of θ but its value remains to be discovered.

This problem looks a little different than the isothermal and the adiabatic problems which involved only $\sigma^2 = \delta^2 + \lambda^2 / \alpha$. Here we have λ^2 / α in one boundary condition.

Now, to obtain σ^2 let us work our problem on the distorted annulus illustrated in Figure 4-7B. In this case the rod wall, presents no problem; it is the pipe wall, whose surface is displaced. For such a case, the surface of the pipe can be defined as

$$R^2 - 2\varepsilon R \cos\theta = R_{out}^2 - \varepsilon^2 \qquad (4.175)$$

where R is a function of θ and ε and where now $R_o = R_{out}$.

Then, we must solve our eigenvalue problem on the displaced domain by solving a sequence of problems on the reference, symmetric, domain. To determine these problems we expand R and σ^2 in powers of ε and obtain problems in $\upsilon_o, \upsilon_1, \upsilon_2, \ldots$ on the reference domain by the usual methods. Write going to second order, the expansion of R becomes

$$R = R_o + \varepsilon R_1 + \frac{\varepsilon^2}{2} R_2 + \ldots \tag{4.176}$$

where

$$R_o = R_{out}, \quad R_1 = \cos\theta_o, \quad R_2 = -\frac{\sin^2\theta_o}{R_{out}}$$

This is much like before.

Now we might hold δ^2 fixed and perturb λ^2 or we might even hold λ^2 fixed and perturb δ^2. We can find the λ^2's for any δ^2 when the pipe is on-center and we are most interested in the value of δ^2 when $\lambda^2 = 0$. This is $\delta^2_{critical}$.

We begin by not declaring the input and allowing both δ^2 and λ^2 to change as the pipe is displaced. The problems are, on the reference domain: At order ε^o

$$\nabla^2 \upsilon_o + \left(\delta_o^2 + \frac{\lambda_o^2}{\alpha}\right)\upsilon_o = 0 \tag{4.177}$$

where υ_o must satisfy

$$\upsilon_o(r_o = R_{out}) = 0 \tag{4.178}$$

and

$$\int_0^{2\pi} \left.\frac{\partial \upsilon_o}{\partial r_o}\right|_{R_{in}} d\theta_o = \eta\left(-\frac{\lambda_o^2}{\alpha}\right)\upsilon_o\Big|_{R_{in}} \tag{4.179}$$

At order ε^1

$$\nabla^2 \upsilon_1 + \left(\delta_o^2 + \frac{\lambda_o^2}{\alpha}\right)\upsilon_1 + \left(\delta_1^2 + \frac{\lambda_1^2}{\alpha}\right)\upsilon_o = 0 \tag{4.180}$$

where υ_1 must satisfy

$$\upsilon_1(r_o = R_{out}) = -R_1 \left. \frac{d\upsilon_o}{dr_o} \right|_{r_o = R_{out}} \tag{4.181}$$

and

$$\int_0^{2\pi} \left. \frac{\partial \upsilon_1}{\partial r_o} \right|_{R_{in}} d\theta_o = \eta \left(-\frac{\lambda_1^2}{\alpha} \right) \upsilon_o \big|_{R_{in}} + \eta \left(-\frac{\lambda_o^2}{\alpha} \right) \upsilon_1 \big|_{R_{in}} \tag{4.182}$$

At order ε^2

$$\nabla^2 \upsilon_2 + \left(\delta_o^2 + \frac{\lambda_o^2}{\alpha} \right) \upsilon_2 + 2 \left(\delta_1^2 + \frac{\lambda_1^2}{\alpha} \right) \upsilon_1 + \left(\delta_2^2 + \frac{\lambda_2^2}{\alpha} \right) \upsilon_o = 0 \tag{4.183}$$

where υ_2 must satisfy

$$\upsilon_2(r_o = R_{out}) = -2R_1 \left. \frac{d\upsilon_1}{dr_o} \right|_{r_o = R_{out}} - R_1^2 \left. \frac{d^2 \upsilon_o}{dr_o^2} \right|_{r_o = R_{out}} - R_2 \left. \frac{d\upsilon_o}{dr_o} \right|_{r_o = R_{out}} \tag{4.184}$$

and

$$\int_0^{2\pi} \left. \frac{\partial \upsilon_2}{\partial r_o} \right|_{R_{in}} d\theta_o = \eta \left(-\frac{\lambda_2^2}{\alpha} \right) \upsilon_o \big|_{R_{in}} + 2\eta \left(-\frac{\lambda_1^2}{\alpha} \right) \upsilon_1 \big|_{R_{in}} + \eta \left(-\frac{\lambda_o^2}{\alpha} \right) \upsilon_2 \big|_{R_{in}} \tag{4.185}$$

First we are going to try and hold λ^2 fixed and equal to zero whence δ^2 is $\delta^2_{critical}$ and we are going to see if we can find out the effect of shifting the pipe on $\delta^2_{critical}$.

Perturb δ^2 and hold $-\lambda^2 = 0$. The problems to be solved are: At order ε^o

$$\nabla^2 \upsilon_o + \delta_o^2 \upsilon_o = 0 \tag{4.186}$$

where υ_o must satisfy

$$\upsilon_o(r_o = R_{out}) = 0 \tag{4.187}$$

and

$$\int_0^{2\pi} \left.\frac{\partial \upsilon_o}{\partial r_o}\right|_{R_{in}} d\theta_o = 0 \tag{4.188}$$

At order ε^1

$$\nabla^2 \upsilon_1 + \delta_o^2 \upsilon_1 + \delta_1^2 \upsilon_o = 0 \tag{4.189}$$

where υ_1 must satisfy

$$\upsilon_1(r_o = R_{out}) = -R_1 \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o = R_{out}} \tag{4.190}$$

and

$$\int_0^{2\pi} \left.\frac{\partial \upsilon_1}{\partial r_o}\right|_{R_{in}} d\theta_o = 0 \tag{4.191}$$

At order ε^2

$$\nabla^2 \upsilon_2 + \delta_o^2 \upsilon_2 + 2\delta_1^2 \upsilon_1 + \delta_2^2 \upsilon_o = 0 \tag{4.192}$$

where υ_2 must satisfy

$$\upsilon_2(r_o = R_{out}) = -2R_1 \left.\frac{d\upsilon_1}{dr_o}\right|_{r_o = R_{out}} - R_1^2 \left.\frac{d^2\upsilon_o}{dr_o^2}\right|_{r_o = R_{out}} - R_2 \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o = R_{out}} \tag{4.193}$$

and

$$\int_0^{2\pi} \left.\frac{\partial \upsilon_2}{\partial r_o}\right|_{R_{in}} d\theta_o = 0 \tag{4.194}$$

At order ε^o the eigenfunctions take the form

$$\left[AJ_M(r_o\delta_o) + BY_M(r_o\delta_o)\right]\cos M\theta_o \tag{4.195}$$

and

$$\left[AJ_M(r_o\delta_o) + BY_M(r_o\delta_o)\right]\sin M\theta_o \tag{4.196}$$

where $M = 0,1,2,...$ and where A, B and δ_o must be determined by the boundary conditions at $r_o = R_{in}$ and $r_o = R_{out}$.

We see already that we have a problem run into a technical difficulty, not at $M = 0$, but at $M = 1,2,...$ the problem is underspecified. So let us focus our attention on the case $M = 0$. Then at order ε^o our problem is just the adiabatic problem and $\hat{\delta}_o$ is just what we found earlier.

However, the constants A and B are not independent, but one can be found in term of the other. Doing this we write our eigenfunction as

$$\upsilon_o = C\left[-Y_o(R_{out}\delta_o)J_o(r_o\delta_o) + Y_o(r_o\delta_o)J_o(R_{out}\delta_o)\right] \tag{4.197}$$

where C is an undetermined constant. This equation can be written in fully scaled form, that is

$$\upsilon_o = C\left[-Y_o(\frac{R_{out}}{R_{in}}\hat{\delta}_o)J_o(\hat{r}_o\hat{\delta}_o) + Y_o(\hat{r}_o\hat{\delta}_o)J_o(\frac{R_{out}}{R_{in}}\hat{\delta}_o)\right] \tag{4.198}$$

and therefore

$$\frac{d\upsilon_o}{d\hat{r}_o} = C\left[-Y_o(\frac{R_{out}}{R_{in}}\hat{\delta}_o)\frac{dJ_o(\hat{r}_o\hat{\delta}_o)}{d\hat{r}_o} + \frac{dY_o(\hat{r}_o\hat{\delta}_o)}{d\hat{r}_o}J_o(\frac{R_{out}}{R_{in}}\hat{\delta}_o)\right] \tag{4.199}$$

Our next job is to correct δ_o^2 for the displacement of the pipe. Then we will have

$$\delta^2 = \delta_{critical}^2 = \delta_o^2 + \varepsilon\delta_1^2 + \frac{\varepsilon^2}{2}\delta_2^2 + ...$$

Now, we turn to order ε^1 and prove δ_1^2 is zero. To do this, multiply Equation (4.189) by υ_o, multiply Equation (4.186) and subtract the second from the first to get

$$\nabla\cdot(\upsilon_o\nabla\upsilon_1) - \nabla\cdot(\upsilon_1\nabla\upsilon_o) + \delta_1^2\upsilon_o^2 = 0 \tag{4.200}$$

Now integrate Equation (4.200) over the reference domain and use the divergence theorem to get

$$
\begin{aligned}
-\int_{R_{in}}^{R_{out}}\int_{0}^{2\pi}\left(\delta_1^2\upsilon_o^2\right)d\theta_o r_o dr_o &= \int_{0}^{2\pi}R_{out}\left(\vec{n}\cdot(\upsilon_o\nabla\upsilon_1)-\vec{n}\cdot(\upsilon_1\nabla\upsilon_o)\right)d\theta_o\bigg|_{r_o=R_{out}} \\
&\quad+\int_{0}^{2\pi}R_{in}\left(\vec{n}\cdot(\upsilon_o\nabla\upsilon_1)-\vec{n}\cdot(\upsilon_1\nabla\upsilon_o)\right)d\theta_o\bigg|_{r_o=R_{in}}
\end{aligned}
\tag{4.201}
$$

where $\vec{n}$ is the unit outward normal to the annular region, as shown in Figure 4-7B.

Each of the four term on the right-hand side is zero because υ_o is zero at $r_o = R_{out}$, υ_1 is multiple of $\cos\theta_o$ at $r_o = R_{out}$, υ_o and υ_1 are constant at $r_o = R_{in}$ and the integral of both $\left(\partial\upsilon_o/\partial r_o\right)$ and $\left(\partial\upsilon_1/\partial r_o\right)$ over $0, 2\pi$ is zero at $r_o = R_{in}$. Hence $\delta_1^2 = 0$ and υ_1 must be found by solving

$$
\nabla^2\upsilon_1 + \delta_o^2\upsilon_1 = 0 \tag{4.202}
$$

where υ_1 is subject to the following two boundary conditions

$$
\upsilon_1(r_o = R_{out}) = -R_1\frac{d\upsilon_o}{dr_o}\bigg|_{r_o=R_{out}} \tag{4.203}
$$

and

$$
\int_{0}^{2\pi}\frac{\partial\upsilon_1}{\partial r_o}\bigg|_{R_{in}} d\theta_o = 0 \tag{4.204}
$$

Now notice that the υ_1 problem is not an eigenvalue problem since δ_o^2 is known, so too $-R_1\left(d\upsilon_o/dr_o\right)\big|_{r_o=R_{out}}$, hence $\upsilon_1 = \upsilon_{1_{Homogeneous}} + \upsilon_{1_{Particular}}$ where

$$
\upsilon_{1_{Homogeneous}} = \dot{C}\left[-Y_o(R_{out}\delta_o)J_o(r_o\delta_o) + Y_o(r_o\delta_o)J_o(R_{out}\delta_o)\right] = \dot{C}\upsilon_o \tag{4.205}
$$

and, because R_1 is a multiple of $\cos\theta_o$, we can get a particular solution in the form

$$\upsilon_{1_{Particular}} = f\left(r_o\right)\cos\theta_o \tag{4.206}$$

whence

$$\upsilon_{1_{Particular}} = \cos\theta_o\left[C_1 J_1(r_o\delta_o) + C_2 Y_1(r_o\delta_o)\right] \tag{4.207}$$

and

$$\upsilon_1 = \dot{C}\upsilon_o + \cos\theta_o\left[C_1 J_1(r_o\delta_o) + C_2 Y_1(r_o\delta_o)\right] \tag{4.208}$$

where the constants C_1 and C_2 must be determined.

Now υ_o satisfies the boundary conditions and hence we would hope to determine C_1 and C_2 using the two conditions on υ_1. But the condition at the rod is satisfied for all C_1 and C_2 hence only the relation between C_1 and C_2 can be established using the condition at the pipe. Doing so, υ_1 will turn out to be

$$\upsilon_1 = \dot{C}\upsilon_o + \left[-\frac{J_1(r_o\delta_o)}{J_1(R_{out}\delta_o)} \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{out}} + C_2\left[-\frac{Y_1(R_{out}\delta_o)}{J_1(R_{out}\delta_o)} J_1(r_o\delta_o) + Y_1(r_o\delta_o)\right]\right]\cos\theta_o \tag{4.209}$$

and $d\upsilon_1/dr_o$, needed later, is given by

$$\frac{d\upsilon_1}{dr_o} = \dot{C}\frac{d\upsilon_o}{dr_o} + \left[-\frac{\frac{dJ_1(r_o\delta_o)}{dr_o}}{J_1(R_{out}\delta_o)} \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{out}} + C_2\varpi_P\right]\cos\theta_o \tag{4.210}$$

where

$$\varpi_P = \left[\frac{-Y_1(R_{out}\delta_o)\frac{dJ_1(r_o\delta_o)}{dr_o} + \frac{dY_1(r_o\delta_o)}{dr_o}J_1(R_{out}\delta_o)}{J_1(R_{out}\delta_o)}\right]$$

unless C_2 disappears, we are not going to be able to find σ_2^2. But let us see.

Again, the homogeneous part will play no role and we drop it henceforth.

The problem for order ε^2 is

$$\nabla^2 \upsilon_2 + \delta_o^2 \upsilon_2 + \delta_2^2 \upsilon_o = 0 \tag{4.211}$$

$$\upsilon_2(r_o = R_{out}) = -2R_1 \left.\frac{d\upsilon_1}{dr_o}\right|_{r_o = R_{out}} - R_1^2 \left.\frac{d^2\upsilon_o}{dr_o^2}\right|_{r_o = R_{out}} - R_2 \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o = R_{out}} \tag{4.212}$$

and

$$\int_0^{2\pi} \left.\frac{\partial \upsilon_2}{\partial r_o}\right|_{R_{in}} d\theta_o = 0 \tag{4.213}$$

Now to get δ_2^2 without determining υ_2 multiply Equation (4.211) by υ_o, multiply Equation (4.186) by υ_2 and subtract the second from the first to get

$$\nabla \cdot (\upsilon_o \nabla \upsilon_2) - \nabla \cdot (\upsilon_2 \nabla \upsilon_o) + \delta_2^2 \upsilon_o^2 = 0 \tag{4.214}$$

Now integrate Equation (4.214) over the reference domain and use the divergence theorem to obtain

$$\begin{aligned} -\int_{R_{in}}^{R_{out}} \int_0^{2\pi} \left(\delta_2^2 \upsilon_o^2\right) d\theta_o r_o dr_o = {} & \int_0^{2\pi} R_{out} \left(\vec{n} \cdot (\upsilon_o \nabla \upsilon_2) - \vec{n} \cdot (\upsilon_2 \nabla \upsilon_o)\right) d\theta_o \Big|_{r_o = R_{out}} \\ & + \int_0^{2\pi} R_{in} \left(\vec{n} \cdot (\upsilon_o \nabla \upsilon_2) - \vec{n} \cdot (\upsilon_2 \nabla \upsilon_o)\right) d\theta_o \Big|_{r_o = R_{in}} \end{aligned} \tag{4.215}$$

Then, because υ_o vanishes at $r_o = R_{out}$, υ_o and υ_1 are constant at $r_o = R_{in}$ and because $\int_0^{2\pi} (d\upsilon_2 / dr_o)\big|_{R_{in}} d\theta_o = 0$ and $\int_0^{2\pi} (d\upsilon_o / dr_o)\big|_{R_{in}} d\theta_o = 0$, Equation (4.215) reduces to

$$\delta_2^2 \int_{R_{in}}^{R_{out}} \int_0^{2\pi} \upsilon_o^2 d\theta_o r_o dr_o = R_{out} \int_0^{2\pi} \upsilon_2\big|_{r_o = R_{out}} \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o = R_{out}} d\theta_o \tag{4.216}$$

where υ_2 is satisfied at R_{out}, therefore

$$\delta_2^2 \int_{R_{in}}^{R_{out}} \int_0^{2\pi} \upsilon_o^2 \, d\theta_o r_o dr_o = -R_{out} \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{out}} \int_0^{2\pi} \left(2R_1 \left.\frac{d\upsilon_1}{dr_o}\right|_{r_o=R_{out}} + R_1^2 \left.\frac{d^2\upsilon_o}{dr_o^2}\right|_{r_o=R_{out}} + R_2 \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{out}} \right) d\theta_o \tag{4.217}$$

Now, using

$$\left.\frac{d^2\upsilon_o}{dr_o^2}\right|_{r_o=R_{out}} = -\frac{1}{R_{out}} \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{out}} \tag{4.218}$$

and Equation (4.210), at $r_o = R_{out}$, without the homogeneous part, viz.,

$$\left.\frac{d\upsilon_1}{dr_o}\right|_{r_o=R_{out}} = \left[-\frac{\left.\frac{dJ_1(r_o\delta_o)}{dr_o}\right|_{r_o=R_{out}}}{J_1(R_{out}\delta_o)} \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{out}} + C_2 \, \bar{\omega}_P\big|_{r_o=R_{out}} \right] \cos\theta_o$$

the formula for δ_2^2 reduces to

$$\delta_2^2 = \frac{R_{out} \frac{\left.\frac{dJ_1(r_o\delta_o)}{dr_o}\right|_{r_o=R_{out}}}{J_1(R_{out}\delta_o)} \left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{out}} - C_2 \, R_{out} \bar{\omega}_P\big|_{r_o=R_{out}} \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{out}} - \left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{out}}}{\int_{R_{in}}^{R_{out}} \upsilon_o^2 r_o dr_o} \tag{4.219}$$

But we do not know C_2. Thus, the perturbation that we had loved does not work.

There are two ways by which we can solve this problem without facing this difficulty. The first way is to solve the problem holding δ^2 fixed and perturb $-\lambda^2/\alpha$, which now becomes our eigenvalue, and we must check for its sign as the rod is shifted. For this case we can choose δ^2 based on our expectation that the critical value of δ^2 in the reference case is equal to that for the adiabatic problem. The second way is to solve the problem holding $-\lambda^2/\alpha$ fixed, at a value other than zero, and perturb δ^2 and see if its value increases or decreases as the rod is shifted. These two ways are illustrated in the following.

Perturb $-\lambda^2/\alpha$ and hold δ^2 fixed. In this case the problems to be solved are: At order ε^o

$$\nabla^2 \upsilon_o + \underbrace{\left(\delta^2 + \frac{\lambda_o^2}{\alpha}\right)}_{\sigma_o^2} \upsilon_o = 0 \tag{4.220}$$

where υ_o must satisfy

$$\upsilon_o(r_o = R_{out}) = 0 \tag{4.221}$$

and

$$\int_0^{2\pi} \left. \frac{\partial \upsilon_o}{\partial r_o} \right|_{R_{in}} d\theta_o = \eta \left(-\frac{\lambda_o^2}{\alpha}\right) \upsilon_o\Big|_{R_{in}} \tag{4.222}$$

At order ε^1

$$\nabla^2 \upsilon_1 + \delta^2 \upsilon_1 + \frac{\lambda_o^2}{\alpha} \upsilon_1 + \frac{\lambda_1^2}{\alpha} \upsilon_o = 0 \tag{4.223}$$

where υ_1 must satisfy

$$\upsilon_1(r_o = R_{out}) = -R_1 \left. \frac{d\upsilon_o}{dr_o} \right|_{r_o = R_{out}} \tag{4.224}$$

and

$$\int_0^{2\pi} \left. \frac{\partial \upsilon_1}{\partial r_o} \right|_{R_{in}} d\theta_o = \eta \left(-\frac{\lambda_1^2}{\alpha}\right) \upsilon_o\Big|_{R_{in}} + \eta \left(-\frac{\lambda_o^2}{\alpha}\right) \upsilon_1\Big|_{R_{in}} \tag{4.225}$$

At order ε^2

$$\nabla^2 \upsilon_2 + \delta^2 \upsilon_2 + \frac{\lambda_o^2}{\alpha} \upsilon_2 + 2\frac{\lambda_1^2}{\alpha} \upsilon_1 + \frac{\lambda_2^2}{\alpha} \upsilon_o = 0 \tag{4.226}$$

where υ_2 must satisfy

$$\upsilon_2(r_o = R_{out}) = -2R_1 \left.\frac{d\upsilon_1}{dr_o}\right|_{r_o=R_{out}} - R_1^2 \left.\frac{d^2\upsilon_o}{dr_o^2}\right|_{r_o=R_{out}} - R_2 \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{out}} \tag{4.227}$$

and

$$\int_0^{2\pi} \left.\frac{\partial \upsilon_2}{\partial r_o}\right|_{R_{in}} d\theta_o = \eta\left(-\frac{\lambda_2^2}{\alpha}\right)\upsilon_o\big|_{R_{in}} + 2\eta\left(-\frac{\lambda_1^2}{\alpha}\right)\upsilon_1\big|_{R_{in}} + \eta\left(-\frac{\lambda_o^2}{\alpha}\right)\upsilon_2\big|_{R_{in}} \tag{4.228}$$

Again, at order ε^o the eigenfunctions take the form

$$\left[AJ_M(r_o\sigma_o) + BY_M(r_o\sigma_o)\right]\cos M\theta_o \tag{4.229}$$

and

$$\left[AJ_M(r_o\sigma_o) + BY_M(r_o\sigma_o)\right]\sin M\theta_o \tag{4.230}$$

where $M = 0,1,2,...$ and where A, B and σ_o must be determined by the boundary conditions at $r_o = R_{in}$ and $r_o = R_{out}$.

To find A and B we turn to Equations (4.221) and (4.222) and obtain

$$AJ_M(R_{out}\sigma_o) + BY_M(R_{out}\sigma_o) = 0 \tag{4.231}$$

and

$$\begin{aligned}\int_0^{2\pi}\left[A\left.\frac{dJ_M(r_o\sigma_o)}{dr_o}\right|_{R_{in}} + B\left.\frac{dY_M(r_o\sigma_o)}{dr_o}\right|_{R_{in}}\right]\cos M\theta_o d\theta_o = \\ \eta\left(-\frac{\lambda_o^2}{\alpha}\right)\left[AJ_M(R_{in}\sigma_o) + BY_M(R_{in}\sigma_o)\right]\cos M\theta_o\end{aligned} \tag{4.232}$$

where the integral on the left-hand side is given by

$$\left[A\left.\frac{dJ_M(r_o\sigma_o)}{dr_o}\right|_{R_{in}} + B\left.\frac{dY_M(r_o\sigma_o)}{dr_o}\right|_{R_{in}}\right]\int_0^{2\pi}\cos M\theta_o d\theta_o \tag{4.233}$$

and this is zero unless $M = 0$. Hence for all M other than zero no heat will be conducted into the rod and because of this the temperature of the rod remains fixed. Consequently the problem is just the problem for constant temperature walls (*i.e.*, isothermal walls).

For $M = 0$ things are little different and this case is our focus. Hence at $M = 0$ we have at $r_o = R_{out}$

$$AJ_o(R_{out}\sigma_o) + BY_o(R_{out}\sigma_o) = 0 \tag{4.234}$$

and at $r_o = R_{in}$

$$A\left.\frac{dJ_o(r_o\sigma_o)}{dr_o}\right|_{R_{in}} + B\left.\frac{dY_o(r_o\sigma_o)}{dr_o}\right|_{R_{in}} = \frac{\eta}{2\pi}\left(-\frac{\lambda_o^2}{\alpha}\right)\left[AJ_o(R_{in}\sigma_o) + BY_o(R_{in}\sigma_o)\right] \tag{4.235}$$

and these two equations can be written as

$$\begin{bmatrix} J_o(R_{out}\sigma_o) & Y_o(R_{out}\sigma_o) \\ \left.\frac{dJ_o(r_o\sigma_o)}{dr_o}\right|_{R_{in}} + \frac{\eta}{2\pi}\left(\frac{\lambda_o^2}{\alpha}\right)J_o(R_{in}\sigma_o) & \left.\frac{dY_o(r_o\sigma_o)}{dr_o}\right|_{R_{in}} + \frac{\eta}{2\pi}\left(\frac{\lambda_o^2}{\alpha}\right)Y_o(R_{in}\sigma_o) \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.236}$$

To get a solution other than $A = 0$ and $B = 0$ the determinant of this matrix must vanish, which means

$$\begin{aligned} J_o(R_{out}\sigma_o)&\left[\left.\frac{dY_o(r_o\sigma_o)}{dr_o}\right|_{R_{in}} + \frac{\eta}{2\pi}\left(\frac{\lambda_o^2}{\alpha}\right)Y_o(R_{in}\sigma_o)\right] \\ &- Y_o(R_{out}\sigma_o)\left[\left.\frac{dJ_o(r_o\sigma_o)}{dr_o}\right|_{R_{in}} + \frac{\eta}{2\pi}\left(\frac{\lambda_o^2}{\alpha}\right)J_o(R_{in}\sigma_o)\right] = 0 \end{aligned} \tag{4.237}$$

But we know that

$$\sigma_o^2 = \left(\delta^2 + \frac{\lambda_o^2}{\alpha}\right) \tag{4.238}$$

then, we can write

$$\frac{\lambda_o^2}{\alpha} = \sigma_o^2 - \delta^2 \tag{4.239}$$

Therefore, by substituting this into Equation (4.237) we get

$$\begin{aligned} J_o(R_{out}\sigma_o)\left[\left.\frac{dY_o(r_o\sigma_o)}{dr_o}\right|_{R_{in}} + \frac{\eta}{2\pi}\left(\sigma_o^2 - \delta^2\right)Y_o(R_{in}\sigma_o)\right] & \\ - Y_o(R_{out}\sigma_o)\left[\left.\frac{dJ_o(r_o\sigma_o)}{dr_o}\right|_{R_{in}} + \frac{\eta}{2\pi}\left(\sigma_o^2 - \delta^2\right)J_o(R_{in}\sigma_o)\right] &= 0 \end{aligned} \tag{4.240}$$

and in fully scaled form

$$\begin{aligned} J_o(\frac{R_{out}}{R_{in}}\hat{\sigma}_o)\left[\left.\frac{dY_o(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o}\right|_{\hat{r}_o=1} + \frac{\hat{\eta}}{2\pi}\left(\hat{\sigma}_o^2 - \hat{\delta}^2\right)Y_o(\hat{\sigma}_o)\right] & \\ - Y_o(\frac{R_{out}}{R_{in}}\hat{\sigma}_o)\left[\left.\frac{dJ_o(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o}\right|_{\hat{r}_o=1} + \frac{\hat{\eta}}{2\pi}\left(\hat{\sigma}_o^2 - \hat{\delta}^2\right)J_o(\hat{\sigma}_o)\right] &= 0 \end{aligned} \tag{4.241}$$

where $\hat{\eta} = \eta / R_{in}$. This is the equation determining the allowable values of $\hat{\sigma}_o$ for the reference domain where $\hat{\delta}^2$ is given and where the critical value of $\hat{\delta}$ is expected to be equal to the critical value of $\hat{\delta}$ in the reference case for the adiabatic inner wall problem (*i.e.*, $\left.\hat{\delta}^2_{critical}\right|_P = \left.\hat{\delta}^2_{critical}\right|_A$), where the subscripts P and A stand for the terms perfectly conducting and adiabatic, respectively.

Solving Equation (4.241) determines the values of $\hat{\sigma}_o$ for any given values of R_{out}/R_{in}, $\hat{\delta}^2$ and $\hat{\eta}$. But because we are interested in obtaining $\hat{\delta}^2_{critical}$, we are going to set $\hat{\delta}^2$ to be equal to $\left.\hat{\delta}^2_{critical}\right|_P$ which is expected to be equal to $\left.\hat{\delta}^2_{critical}\right|_A$. The value of $\left.\hat{\delta}^2_{critical}\right|_A$ for any given value of R_{out}/R_{in} can be obtained from section 4.2.1.2.

To check if this expectation is correct, let us choose a value for R_{out}/R_{in} and obtain the corresponding value of $\hat{\delta}^2_{critical}\big|_A$ from the previous section. Then, let us set values for $\hat{\delta}^2$ such that it is slightly higher than $\hat{\delta}^2_{critical}\big|_A$, equal to $\hat{\delta}^2_{critical}\big|_A$ and slightly lower than $\hat{\delta}^2_{critical}\big|_A$ and see how the eigenvalues, $-\lambda_o^2/\alpha$, behave. To do this, substitute the chosen value of $\hat{\delta}^2$ into Equation (4.241) with a selected value for $\hat{\eta}$ and with the previously selected R_{out}/R_{in}. Some results are presented in Table 4-3.

From our calculation, see Table 4-3, we found that for the case $\hat{\delta}^2 > \hat{\delta}^2_{critical}\big|_A$ the first eigenvalue , $-\left(\lambda_o^2/\alpha\right)_1$, is positive while the remaining eigenvalues, $-\left(\lambda_o^2/\alpha\right)_2$, $-\left(\lambda_o^2/\alpha\right)_3$,... are all negative. From this we can conclude that the selected value of $\hat{\delta}^2$ is greater than $\hat{\delta}^2_{critical}\big|_P$ and, hence, the system is unstable at this value of $\hat{\delta}^2$.

Table 4-3. Values of $\hat{\delta}^2_{critical}\big|_P$ at $\hat{\eta} = 2\pi$ and $R_{out}/R_{in} = 5$.

	$\hat{\delta}^2 > \hat{\delta}^2_{critical}\big\|_A$	$\hat{\delta}^2 = \hat{\delta}^2_{critical}\big\|_A$	$\hat{\delta}^2 < \hat{\delta}^2_{critical}\big\|_A$
$\hat{\delta}^2$	0.270	0.264	0.260
$-\left(\lambda_o^2/\alpha\right)_1$	3.74×10^{-3}	0	-4.2×10^{-3}
$-\left(\lambda_o^2/\alpha\right)_2$	-0.875	-0.880	-0.885
$-\left(\lambda_o^2/\alpha\right)_3$	-2.700	-2.710	-2.720

In the case where $\hat{\delta}^2 = \hat{\delta}^2_{critical}\big|_A$, we found that the first eigenvalue, $-\left(\lambda_o^2/\alpha\right)_1$, is zero while all the other eigenvalues are negative. From this we can conclude that $\hat{\delta}^2$ is equal to $\hat{\delta}^2_{critical}\Big|_P$ and, consequently, $\hat{\delta}^2_{critical}\Big|_P = \hat{\delta}^2_{critical}\Big|_A$ and, therefore, our expectation is correct.

Moreover, in the case $\hat{\delta}^2 < \hat{\delta}^2_{critical}\Big|_A$ we found that all the eigenvalues are negative which indicates that the selected value of $\hat{\delta}^2$ is less than $\hat{\delta}^2_{critical}\Big|_P$ and, hence, the system is stable for all values of $\hat{\delta}^2$ equal to or less than the selected $\hat{\delta}^2$.

Now, because $\hat{\delta}^2_{critical}\Big|_P = \hat{\delta}^2_{critical}\Big|_A$ in the symmetric case and because we know that the critical value of $\hat{\delta}$ is reached when $-\lambda_o^2 = 0$ ($i.e.$, $\hat{\sigma}_o^2 = \hat{\delta}^2 = \hat{\delta}^2_{critical}$), then the value of $\hat{\eta}$ will not affect the first value of $\hat{\sigma}_o$, $\hat{\sigma}_{o_1}$, and, consequently, it will not affect the first value of $-\left(\lambda_o^2/\alpha\right)_1$. This is because, as we can see in Equation (4.241), $\hat{\eta}$ is multiplied by $\left(\hat{\sigma}_o^2 - \hat{\delta}^2\right)$ which is in this case equal to zero and, therefore, Equation (4.241) reduces to

$$J_o\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right)\frac{dY_o(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o}\Bigg|_{\hat{r}_o=1} - \frac{dJ_o(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o}\Bigg|_{\hat{r}_o=1} Y_o\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right) = 0 \qquad (4.242)$$

which is similar to Equation (4.139) obtained from the adiabatic inner wall problem.

However, solving Equation (4.241) determines the values of $\hat{\sigma}_o$ for any given value of R_{out}/R_{in}, $\hat{\delta}$ and $\hat{\eta}$. But because $\hat{\delta}^2$ is set to be equal to $\hat{\delta}^2_{critical}$, $\hat{\delta}^2_{critical}\Big|_P = \hat{\delta}^2_{critical}\Big|_A$, then the value of $\hat{\sigma}_o$ can be affected by two factors, they are R_{out}/R_{in} and $\hat{\eta}$. To study the effect of each of these two factors on the value of $\hat{\sigma}_o$ the calculation will be done as follows: First we will fix $\hat{\eta}$ and determine the values of $\hat{\sigma}_o$ at various values of R_{out}/R_{in}, then we will fix R_{out}/R_{in} and determine the values of $\hat{\sigma}_o$ at various values of $\hat{\eta}$. The results obtained for the two cases are illustrated in Tables 4-4 and 4-5.

Table 4-4. Values of $\hat{\sigma}_o$ at $\hat{\eta}=2\pi$ and various R_{out}/R_{in}.

R_{out}/R_{in}	$\hat{\sigma}_o$
5	0.514
10	0.245
15	0.162

Table 4-5. Lowest values of $\hat{\sigma}_o$ at $R_{out}/R_{in}=5$ and various $\hat{\eta}$.

$\hat{\eta}$	$\hat{\sigma}_o\|_1$	$\hat{\sigma}_o\|_2$
0.1π	0.514	1.99
2π	0.514	1.72
10π	0.514	1.59

From Table 4-4 we can notice that thick heat-producing regions are thermally less stable than thin heat-producing regions. This result agrees with our earlier calculations.

From Table 4-5, we can see that the value of $\hat{\eta}$ will not affect the first value of $\hat{\sigma}_o$, $\hat{\sigma}_{o_1}$, which is the value of interest, and, consequently, it will not affect the first value of $-\left(\lambda_o^2/\alpha\right)_1$ as we explained earlier.

However, the constants A and B are not independent, but one can be found in term of the other. Doing this we write our eigenfunction as

$$\upsilon_o = C\left[-Y_o(R_{out}\sigma_o)J_o(r_o\sigma_o)+Y_o(r_o\sigma_o)J_o(R_{out}\sigma_o)\right] \tag{4.243}$$

where C is an undetermined constant. This equation can be written in fully scaled form, that is

$$\upsilon_o = C\left[-Y_o(\frac{R_{out}}{R_{in}}\hat{\sigma}_o)J_o(\hat{r}_o\hat{\sigma}_o)+Y_o(\hat{r}_o\hat{\sigma}_o)J_o(\frac{R_{out}}{R_{in}}\hat{\sigma}_o)\right] \tag{4.244}$$

and therefore

$$\frac{d\upsilon_o}{d\hat{r}_o} = C\left[-Y_o(\frac{R_{out}}{R_{in}}\hat{\sigma}_o)\frac{dJ_o(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o} + \frac{dY_o(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o}J_o(\frac{R_{out}}{R_{in}}\hat{\sigma}_o)\right] \tag{4.245}$$

Having the value of $\hat{\sigma}_o^2$, the values of $-\lambda_o^2/\alpha$ can be determined via

$$-\frac{\lambda_o^2}{\alpha} = \hat{\delta}^2 - \hat{\sigma}_o^2$$

Our next job is to correct $-\lambda_o^2/\alpha$ for the displacement of the pipe. Then we will have σ^2 as

$$\sigma^2 = \left(\delta^2 + \frac{\lambda^2}{\alpha}\right) = \left(\delta^2 + \frac{\lambda_o^2}{\alpha}\right) + \varepsilon\left(\frac{\lambda_1^2}{\alpha}\right) + \frac{\varepsilon^2}{2}\left(\frac{\lambda_2^2}{\alpha}\right) + \dots$$

Now, to find $-\lambda_1^2/\alpha$ we must solve Equation (4.223), which is

$$\nabla^2\upsilon_1 + \frac{\lambda_1^2}{\alpha}\upsilon_o + \delta^2\upsilon_1 + \frac{\lambda_o^2}{\alpha}\upsilon_1 = 0$$

where υ_1 must satisfy

$$\upsilon_1(r_o = R_{out}) = -R_1\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{out}}$$

and

$$\int_0^{2\pi}\left.\frac{\partial\upsilon_1}{\partial r_o}\right|_{R_{in}} d\theta_o = \eta\left(-\frac{\lambda_1^2}{\alpha}\right)\upsilon_o\big|_{R_{in}} + \eta\left(-\frac{\lambda_o^2}{\alpha}\right)\upsilon_1\big|_{R_{in}}$$

To obtain $-\lambda_1^2/\alpha$ let us multiply Equation (4.223) by υ_o and Equation (4.220) by υ_1 and subtract the second from the first to obtain

$$\nabla\cdot(\upsilon_o\nabla\upsilon_1) - \nabla\cdot(\upsilon_1\nabla\upsilon_o) + \frac{\lambda_1^2}{\alpha}\upsilon_o^2 = 0 \tag{4.246}$$

Now integrate Equation (4.246) over the reference domain and use the divergence theorem to get

$$-\int_{R_{in}}^{R_{out}}\int_{0}^{2\pi}\left(\frac{\lambda_1^2}{\alpha}\upsilon_o^2\right)d\theta_o r_o dr_o = \int_0^{2\pi} R_{out}\left(\vec{n}\cdot(\upsilon_o\nabla\upsilon_1)-\vec{n}\cdot(\upsilon_1\nabla\upsilon_o)\right)d\theta_o\Big|_{r_o=R_{out}} + \int_0^{2\pi} R_{in}\left(\vec{n}\cdot(\upsilon_o\nabla\upsilon_1)-\vec{n}\cdot(\upsilon_1\nabla\upsilon_o)\right)d\theta_o\Big|_{r_o=R_{in}} \tag{4.247}$$

where $\vec{n}$ is the unit outward normal to the annular region, as shown in Figure 4-7B.

Now the first integral in the right-hand side of Equation (4.247) is zero due to $\upsilon_o(r_o = R_{out}) = 0$ and because $\upsilon_1(r_o = R_{out})$ is a multiple of $\cos\theta_o$ and when integrated over $0, 2\pi$ it vanishes. Hence we need to pay attention only to the second integral, hence this equation becomes

$$\left(\frac{\lambda_1^2}{\alpha}\right)\int_{R_{in}}^{R_{out}}\int_0^{2\pi}\upsilon_o^2 d\theta_o r_o dr_o = R_{in}\,\upsilon_o\big|_{r_o=R_{in}}\int_0^{2\pi}\frac{d\upsilon_1}{dr_o}\Big|_{r_o=R_{in}} d\theta_o - R_{in}\,\upsilon_1\big|_{r_o=R_{in}}\int_0^{2\pi}\frac{d\upsilon_o}{dr_o}\Big|_{r_o=R_{in}} d\theta_o \tag{4.248}$$

By using Equations (4.222) and (4.225), Equation (4.248) becomes

$$\left(\frac{\lambda_1^2}{\alpha}\right)\left[\int_{R_{in}}^{R_{out}}\int_0^{2\pi}\upsilon_o^2 d\theta_o r_o dr_o + R_{in}\eta\upsilon_o^2\big|_{R_{in}}\right] = 0 \tag{4.249}$$

Because υ_o^2 is neither zero nor negative and η cannot be negative, then $-\lambda_1^2/\alpha$ must be zero. Again, this is an expected result and it happened because of the symmetry of the annulus. Hence to learn the effect of a displacement we must go to second order of ε. But there we will need υ_1 which can be determined by solving

$$\nabla^2\upsilon_1 + \delta^2\upsilon_1 + \frac{\lambda_o^2}{\alpha}\upsilon_1 = 0 \tag{4.250}$$

which can be written as

$$\nabla^2\upsilon_1 + \sigma_o^2\upsilon_1 = 0 \tag{4.251}$$

where

$$\upsilon_1(r_o = R_{out}) = -R_1 \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{out}} \tag{4.252}$$

and

$$\int_0^{2\pi} \left.\frac{\partial \upsilon_1}{\partial r_o}\right|_{R_{in}} d\theta_o = \eta\left(-\frac{\lambda_o^2}{\alpha}\right)\upsilon_1\big|_{R_{in}} \tag{4.253}$$

Notice that the υ_1 problem is not an eigenvalue problem since σ_o^2 is known and its general solution is the sum of a homogeneous solution where $-R_1\left(d\upsilon_o/dr_o\right)\big|_{r_o=R_{out}}$ is set to zero and a particular solution. The homogeneous part is

$$\upsilon_{1_{Homogeneous}} = \dot{C}\left[-Y_o(R_{out}\sigma_o)J_o(r_o\sigma_o) + Y_o(r_o\sigma_o)J_o(R_{out}\sigma_o)\right] = \dot{C}\upsilon_o \tag{4.254}$$

and, because R_1 is a multiple of $\cos\theta_o$, we can get a particular solution in the form

$$\upsilon_{1_{Particular}} = f\left(r_o\right)\cos\theta_o \tag{4.255}$$

Now, by substituting Equation (4.255) into Equation (4.251) and solving for $f\left(r_o\right)$, a particular solution to the υ_1 problem can be written as

$$\upsilon_{1_{Particular}} = \cos\theta_o\left[C_1J_1(r_o\sigma_o) + C_2Y_1(r_o\sigma_o)\right] \tag{4.256}$$

and

$$\upsilon_1 = \dot{C}\upsilon_o + \cos\theta_o\left[C_1J_1(r_o\sigma_o) + C_2Y_1(r_o\sigma_o)\right] \tag{4.257}$$

where the constants C_1 and C_2 can be determined by using Equations (4.252) and (4.253). Doing this, υ_1 will turn out to be

$$\upsilon_1 = \dot{C}\upsilon_o + \cos\theta_o\left[\frac{Y_1(R_{in}\sigma_o)J_1(r_o\sigma_o) - Y_1(r_o\sigma_o)J_1(R_{in}\sigma_o)}{-Y_1(R_{in}\sigma_o)J_1(R_{out}\sigma_o) + Y_1(R_{out}\sigma_o)J_1(R_{in}\sigma_o)}\right]\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{out}} \tag{4.258}$$

and $d\upsilon_1/dr_o$, needed later, is given by

$$\frac{d\upsilon_1}{dr_o} = \dot{C}\frac{d\upsilon_o}{dr_o} + \cos\theta_o \omega_P \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{out}} \tag{4.259}$$

where

$$\omega_P = \left[\frac{Y_1(R_{in}\sigma_o)\dfrac{dJ_1(r_o\sigma_o)}{dr_o} - \dfrac{dY_1(r_o\sigma_o)}{dr_o}J_1(R_{in}\sigma_o)}{-Y_1(R_{in}\sigma_o)J_1(R_{out}\sigma_o) + Y_1(R_{out}\sigma_o)J_1(R_{in}\sigma_o)}\right]$$

Again, the homogeneous part will play no role and we drop it henceforth.

The problem at order ε^2 is

$$\nabla^2\upsilon_2 + \delta^2\upsilon_2 + \frac{\lambda_o^2}{\alpha}\upsilon_2 + \frac{\lambda_2^2}{\alpha}\upsilon_o = 0 \tag{4.260}$$

$$\upsilon_2(r_o = R_{out}) = -2R_1\left.\frac{d\upsilon_1}{dr_o}\right|_{r_o=R_{out}} - R_1^2\left.\frac{d^2\upsilon_o}{dr_o^2}\right|_{r_o=R_{out}} - R_2\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=R_{out}} \tag{4.261}$$

and

$$\int_0^{2\pi}\left.\frac{\partial\upsilon_2}{\partial r_o}\right|_{R_{in}} d\theta_o = \eta\left(-\frac{\lambda_2^2}{\alpha}\right)\upsilon_o\big|_{R_{in}} + \eta\left(-\frac{\lambda_o^2}{\alpha}\right)\upsilon_2\big|_{R_{in}} \tag{4.262}$$

Now to get $-\lambda_2^2/\alpha$ without determining υ_2 multiply Equation (4.260) by υ_o, multiply Equation (4.220) by υ_2 and subtract the second from the first to obtain

$$\nabla\cdot\left(\upsilon_o\nabla\upsilon_2\right) - \nabla\cdot\left(\upsilon_2\nabla\upsilon_o\right) + \frac{\lambda_2^2}{\alpha}\upsilon_o^2 = 0 \tag{4.263}$$

Now integrate Equation (4.263) over the reference domain and use the divergence theorem to get

$$\begin{aligned}-\int_{R_{in}}^{R_{out}}\int_{0}^{2\pi}\left(\frac{\lambda_2^2}{\alpha}\upsilon_o^2\right)d\theta_o r_o dr_o &= \int_{0}^{2\pi} R_{out}\left(\vec{n}\cdot\left(\upsilon_o\nabla\upsilon_2\right)-\vec{n}\cdot\left(\upsilon_2\nabla\upsilon_o\right)\right)d\theta_o\Bigg|_{r_o=R_{out}} \\ &\quad+\int_{0}^{2\pi} R_{in}\left(\vec{n}\cdot\left(\upsilon_o\nabla\upsilon_2\right)-\vec{n}\cdot\left(\upsilon_2\nabla\upsilon_o\right)\right)d\theta_o\Bigg|_{r_o=R_{in}}\end{aligned} \tag{4.264}$$

Then, because υ_o vanishes at $r_o = R_{out}$ Equation (4.264) becomes

$$\begin{aligned}-\left(\frac{\lambda_2^2}{\alpha}\right)\int_{R_{in}}^{R_{out}}\int_{0}^{2\pi}\upsilon_o^2 d\theta_o r_o dr_o &= -R_{out}\int_{0}^{2\pi}\upsilon_2\big|_{r_o=R_{out}}\frac{d\upsilon_o}{dr_o}\bigg|_{r_o=R_{out}}d\theta_o \\ &\quad -R_{in}\int_{0}^{2\pi}\upsilon_o\big|_{r_o=R_{in}}\frac{d\upsilon_2}{dr_o}\bigg|_{r_o=R_{in}}d\theta_o + R_{in}\int_{0}^{2\pi}\upsilon_2\big|_{r_o=R_{in}}\frac{d\upsilon_o}{dr_o}\bigg|_{r_o=R_{in}}d\theta_o\end{aligned} \tag{4.265}$$

and because $d\upsilon_o/dr_o$ is independent of θ_o and $\upsilon_o(r_o = R_{in})$ and $\upsilon_1(r_o = R_{in})$ are fixed and by using Equations (4.222), (4.261) and (4.262), Equation (4.264) reduces to

$$\begin{aligned}-\left(\frac{\lambda_2^2}{\alpha}\right)\int_{R_{in}}^{R_{out}}\int_{0}^{2\pi}\upsilon_o^2 d\theta_o r_o dr_o &= R_{out}\frac{d\upsilon_o}{dr_o}\bigg|_{r_o=R_{out}}\int_{0}^{2\pi}\left(2R_1\frac{d\upsilon_1}{dr_o}\bigg|_{r_o=R_{out}}+R_1^2\frac{d^2\upsilon_o}{dr_o^2}\bigg|_{r_o=R_{out}}\right. \\ &\quad\left.+R_2\frac{d\upsilon_o}{dr_o}\bigg|_{r_o=R_{out}}\right)d\theta_o+\left(\frac{\lambda_2^2}{\alpha}\right)\eta R_{in}\,\upsilon_o^2\big|_{r_o=R_{in}}\end{aligned} \tag{4.266}$$

Now, using

$$\frac{d^2\upsilon_o}{dr_o^2}\bigg|_{r_o=R_{out}} = -\frac{1}{R_{out}}\frac{d\upsilon_o}{dr_o}\bigg|_{r_o=R_{out}} \tag{4.267}$$

and using Equation (4.259), at $r_o = R_{out}$, without the homogeneous part, viz.,

$$\frac{d\upsilon_1}{dr_o}\bigg|_{r_o=R_{out}} = \cos\theta_o\,\omega_P\big|_{r_o=R_{out}}\frac{d\upsilon_o}{dr_o}\bigg|_{r_o=R_{out}}$$

the formula for $-\lambda_2^2/\alpha$ reduces to

$$-\frac{\lambda_2^2}{\alpha}=\frac{\left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{out}}\int\limits_0^{2\pi}\left(2R_{out}\,\omega_P\big|_{r_o=R_{out}}\cos^2\theta_o-1\right)d\theta_o}{2\pi\int\limits_{R_{in}}^{R_{out}}\upsilon_o^2 r_o dr_o+\eta R_{in}^2\,\upsilon_o^2\big|_{r_o=R_{in}}} \tag{4.268}$$

Notice that the denominator of this equation is always positive.

On evaluating the integrals in the numerator and the denominator we obtain

$$-\frac{\lambda_2^2}{\alpha}=\frac{\left(R_{out}\,\omega_P\big|_{r_o=R_{out}}-1\right)\left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{out}}}{\left[\frac{R_{out}^2}{2\sigma_o^2}\left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{out}}-\frac{R_{in}^2}{2\sigma_o^2}\left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{in}}-\frac{R_{in}^2}{2}\upsilon_o^2\Big|_{r_o=R_{in}}+\left(\frac{\eta}{2\pi}\right)R_{in}^2\upsilon_o^2\Big|_{r_o=R_{in}}\right]} \tag{4.269}$$

and in fully scaled form it becomes

$$-\frac{\hat{\lambda}_2^2}{\hat{\alpha}}=\frac{\left(\hat{\omega}_P\big|_{\hat{r}_o=\frac{R_{out}}{R_{in}}}-1\right)\left(\frac{d\upsilon_o}{d\hat{r}_o}\right)^2_{\hat{r}_o=\frac{R_{out}}{R_{in}}}}{\left[\left(\frac{R_{out}}{R_{in}}\right)^2\frac{1}{2\hat{\sigma}_o^2}\left(\frac{d\upsilon_o}{d\hat{r}_o}\right)^2_{\hat{r}_o=\frac{R_{out}}{R_{in}}}-\frac{1}{2\hat{\sigma}_o^2}\left(\frac{d\upsilon_o}{d\hat{r}_o}\right)^2_{\hat{r}_o=1}-\frac{1}{2}\upsilon_o^2\Big|_{\hat{r}_o=1}+\left(\frac{\hat{\eta}}{2\pi}\right)\upsilon_o^2\Big|_{\hat{r}_o=1}\right]} \tag{4.270}$$

where

$$\frac{d\upsilon_o}{d\hat{r}_o}=C\left[-Y_o\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right)\frac{dJ_o(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o}+\frac{dY_o(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o}J_o\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right)\right]$$

and

$$\hat{\omega}_P=\frac{R_{out}}{R_{in}}\left[\frac{Y_1(\hat{\sigma}_o)\frac{dJ_1(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o}-\frac{dY_1(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o}J_1(\hat{\sigma}_o)}{-Y_1(\hat{\sigma}_o)J_1\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right)+Y_1\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right)J_1(\hat{\sigma}_o)}\right]$$

Notice that $-\hat{\lambda}_2^2/\hat{\alpha}$ depends on $\hat{r}_o$, whose value can range from one to R_{out}/R_{in}, and on $\hat{\eta}$, whose value can range from zero to infinity.

Now the sign of $-\hat{\lambda}_2^2/\hat{\alpha}$ depends only on the sign of the factor $\left(\hat{\omega}_P\big|_{\hat{r}_o=R_{out}/R_{in}}-1\right)$ because the signs of all the other components of Equation (4.270) are known, they are positive. As in the previous two problems, we have been unable to determine the sign of $\left(\hat{\omega}_P\big|_{\hat{r}_o=R_{out}/R_{in}}-1\right)$ without a calculation. But the calculations show that it is negative and, consequently, the sign of $-\lambda_2^2/\alpha$ must be negative.

Thus far all the components of the eigenvalue, $-\lambda^2/\alpha$, are obtained (*i.e.*, $-\lambda_o^2/\alpha$, $-\lambda_1^2/\alpha$ and $-\lambda_2^2/\alpha$). The next step is to determine the value of $-\lambda^2/\alpha$ for the distorted annulus, which is defined as

$$-\frac{\lambda^2}{\alpha}=\left(-\frac{\lambda_o^2}{\alpha}\right)+\varepsilon\left(-\frac{\lambda_1^2}{\alpha}\right)+\frac{\varepsilon^2}{2}\left(-\frac{\lambda_2^2}{\alpha}\right)+\dots$$

and then compare its value with that at the reference state (*i.e.*, $-\lambda_o^2/\alpha$). To do this, let us keep in mind that, $-\lambda_2^2/\alpha$ and is a negative quantity, ε is a positive quantity and $-\lambda_1^2/\alpha$ is zero. Then we can notice that the eigenvalue, $-\lambda^2/\hat{\alpha}$, for the distorted annulus is less than it is for the reference state. Whereupon $\hat{\alpha}$ is assumed to be fixed, then this can only mean that the growth constant, $-\lambda^2$, of the distorted annulus is lower than the growth constant of the reference state. And we know that, as the growth constant decreases the thermal stability of the system increases. And because in this case $\hat{\delta}^2$ is hold fixed, then the increase in the thermal stability will be due to an increase in the thermal diffusion. Therefore, we can say that the effects of the θ-diffusion and the heat conduction through the rod overcome the effect of the wider region to the right of the rod.

Based on this we can conclude that, at perfectly conducting inner wall, a distorted annular region is thermally more stable than a symmetric one. Therefore, the perturbation in this case increases the thermal stability of the annular region.

Now, to check our calculation, let us solve this problem by using the second method (*i.e.*, holding $-\lambda^2/\alpha$ fixed at a value other than zero and perturb δ^2), and see if we can recover the same conclusion.

Perturb δ^2 and hold $-\lambda^2/\alpha$ fixed at a value other than zero. In this case the problems to be solved are: At order ε^o the domain equation is

$$\nabla^2 \upsilon_o + \underbrace{\left(\delta_o^2 + \frac{\lambda^2}{\alpha} \right)}_{\sigma_o^2} \upsilon_o = 0 \tag{4.271}$$

where υ_o must satisfy

$$\upsilon_o(r_o = R_{out}) = 0 \tag{4.272}$$

and

$$\int_0^{2\pi} \left. \frac{\partial \upsilon_o}{\partial r_o} \right|_{R_{in}} d\theta_o = \eta \left(-\frac{\lambda^2}{\alpha} \right) \upsilon_o \big|_{R_{in}} \tag{4.273}$$

At order ε^1

$$\nabla^2 \upsilon_1 + \left(\delta_o^2 + \frac{\lambda^2}{\alpha} \right) \upsilon_1 + \delta_1^2 \upsilon_o = 0 \tag{4.274}$$

where υ_1 must satisfy

$$\upsilon_1(r_o = R_{out}) = -R_1 \left. \frac{d\upsilon_o}{dr_o} \right|_{R_{out}} \tag{4.275}$$

and

$$\int_0^{2\pi} \left.\frac{\partial \upsilon_1}{\partial r_o}\right|_{R_{in}} d\theta_o = \eta\left(-\frac{\lambda^2}{\alpha}\right)\upsilon_1\big|_{R_{in}} \tag{4.276}$$

At order ε^2

$$\nabla^2\upsilon_2 + \left(\delta_0^2 + \frac{\lambda^2}{\alpha}\right)\upsilon_2 + 2\delta_1^2\upsilon_1 + \delta_2^2\upsilon_o = 0 \tag{4.277}$$

where υ_2 must satisfy

$$\upsilon_2(r_o = R_{out}) = -2R_1\left.\frac{d\upsilon_1}{dr_o}\right|_{R_{out}} - R_1^2\left.\frac{d^2\upsilon_o}{dr_o^2}\right|_{R_{out}} - R_2\left.\frac{d\upsilon_o}{dr_o}\right|_{R_{out}} \tag{4.278}$$

and

$$\int_0^{2\pi} \left.\frac{\partial \upsilon_2}{\partial r_o}\right|_{R_{in}} d\theta_o = \eta\left(-\frac{\lambda^2}{\alpha}\right)\upsilon_2\big|_{R_{in}} \tag{4.279}$$

Again, at order ε^o the eigenfunctions take the form

$$\left[AJ_M(r_o\sigma_o) + BY_M(r_o\sigma_o)\right]\cos M\theta_o \tag{4.280}$$

and

$$\left[AJ_M(r_o\sigma_o) + BY_M(r_o\sigma_o)\right]\sin M\theta_o \tag{4.281}$$

where $M = 0,1,2,...$ and where A, B and σ_o must be determined by the boundary conditions at $r_o = R_{in}$ and $r_o = R_{out}$.

To find A and B we turn to Equations (4.272) and (4.273) and obtain

$$AJ_M(R_{out}\sigma_o) + BY_M(R_{out}\sigma_o) = 0 \tag{4.282}$$

and

$$\int_0^{2\pi}\left[A\left.\frac{dJ_M(r_o\sigma_o)}{dr_o}\right|_{R_{in}}+B\left.\frac{dY_M(r_o\sigma_o)}{dr_o}\right|_{R_{in}}\right]\cos M\theta_o d\theta_o = \eta\left(-\frac{\lambda^2}{\alpha}\right)\left[AJ_M(R_{in}\sigma_o)+BY_M(R_{in}\sigma_o)\right]\cos M\theta_o \tag{4.283}$$

where the integral on the left-hand side is given by

$$\left[A\left.\frac{dJ_M(r_o\sigma_o)}{dr_o}\right|_{R_{in}}+B\left.\frac{dY_M(r_o\sigma_o)}{dr_o}\right|_{R_{in}}\right]\int_0^{2\pi}\cos M\theta_o d\theta_o \tag{4.284}$$

and this is zero unless $M=0$. Again, for all M other than zero no heat will be conducted into the rod and because of this the temperature of the rod cannot changes. As a result of this the problem becomes just like the problem of constant temperatures walls.

For $M=0$ things are little different and this case is our focus. Hence at $M=0$ we have at $r_o=R_{out}$

$$AJ_o(R_{out}\sigma_o)+BY_o(R_{out}\sigma_o)=0 \tag{4.285}$$

and at $r_o=R_{in}$

$$A\left.\frac{dJ_o(r_o\sigma_o)}{dr_o}\right|_{R_{in}}+B\left.\frac{dY_o(r_o\sigma_o)}{dr_o}\right|_{R_{in}}=\frac{\eta}{2\pi}\left(-\frac{\lambda^2}{\alpha}\right)\left[AJ_o(R_{in}\sigma_o)+BY_o(R_{in}\sigma_o)\right] \tag{4.286}$$

and these two equations can be written as

$$\begin{bmatrix} J_o(R_{out}\sigma_o) & Y_o(R_{out}\sigma_o) \\ \left.\frac{dJ_o(r_o\sigma_o)}{dr_o}\right|_{R_{in}}+\frac{\eta}{2\pi}\left(\frac{\lambda^2}{\alpha}\right)J_o(R_{in}\sigma_o) & \left.\frac{dY_o(r_o\sigma_o)}{dr_o}\right|_{R_{in}}+\frac{\eta}{2\pi}\left(\frac{\lambda^2}{\alpha}\right)Y_o(R_{in}\sigma_o)\end{bmatrix}\begin{bmatrix}A\\B\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \tag{4.287}$$

To get a solution other than $A=0$ and $B=0$ this determinant must vanish, which means

$$J_o(R_{out}\sigma_o)\left[\left.\frac{dY_o(r_o\sigma_o)}{dr_o}\right|_{R_{in}}+\frac{\eta}{2\pi}\left(\frac{\lambda^2}{\alpha}\right)Y_o(R_{in}\sigma_o)\right]$$
$$-Y_o(R_{out}\sigma_o)\left[\left.\frac{dJ_o(r_o\sigma_o)}{dr_o}\right|_{R_{in}}+\frac{\eta}{2\pi}\left(\frac{\lambda^2}{\alpha}\right)J_o(R_{in}\sigma_o)\right]=0 \tag{4.288}$$

Now, to determine δ_o^2 which indicates the strength of the heat source for the symmetric annulus, we first must decide about the value of $-\lambda^2/\alpha$ that we are going to use. One-way to do this is to use Table 4-3 where the value of δ^2 is known (*i.e.*, $\hat{\delta}^2=\hat{\delta}^2_{critical}\big|_A$), and choose a value of $-\lambda^2/\alpha$ other than zero. Then, in this calculation $\hat{\delta}_o$ is just $\hat{\delta}$ and whereupon σ_o^2 is given by

$$\sigma_o^2=\delta_o^2+\frac{\lambda^2}{\alpha} \tag{4.289}$$

then, we can write

$$\sigma_o=\sqrt{\delta_o^2+\frac{\lambda^2}{\alpha}} \tag{4.290}$$

Therefore, by substituting this into Equation (4.288) we get

$$J_o(R_{out}\sqrt{\delta_o^2+\frac{\lambda^2}{\alpha}})\left[\left.\frac{dY_o(r_o\sqrt{\delta_o^2+\frac{\lambda^2}{\alpha}})}{dr_o}\right|_{R_{in}}+\frac{\eta}{2\pi}\left(\frac{\lambda^2}{\alpha}\right)Y_o(R_{in}\sqrt{\delta_o^2+\frac{\lambda^2}{\alpha}})\right]$$
$$-Y_o(R_{out}\sqrt{\delta_o^2+\frac{\lambda^2}{\alpha}})\left[\left.\frac{dJ_o(r_o\sqrt{\delta_o^2+\frac{\lambda^2}{\alpha}})}{dr_o}\right|_{R_{in}}+\frac{\eta}{2\pi}\left(\frac{\lambda^2}{\alpha}\right)J_o(R_{in}\sqrt{\delta_o^2+\frac{\lambda^2}{\alpha}})\right]=0 \tag{4.291}$$

and in fully scaled form

$$\begin{aligned} & J_o\left(\frac{R_{out}}{R_{in}}\sqrt{\hat{\delta}_o^2+\frac{\lambda^2}{\hat{\alpha}}}\right)\left[\left. \frac{dY_o\left(\hat{r}_o\sqrt{\hat{\delta}_o^2+\frac{\lambda^2}{\hat{\alpha}}}\right)}{d\hat{r}_o}\right|_{\hat{r}_o=1} +\frac{\hat{\eta}}{2\pi}\left(\frac{\lambda^2}{\hat{\alpha}}\right)Y_o\left(\sqrt{\hat{\delta}_o^2+\frac{\lambda^2}{\hat{\alpha}}}\right)\right] \\ & -Y_o\left(\frac{R_{out}}{R_{in}}\sqrt{\hat{\delta}_o^2+\frac{\lambda^2}{\hat{\alpha}}}\right)\left[\left. \frac{dJ_o\left(\hat{r}_o\sqrt{\hat{\delta}_o^2+\frac{\lambda^2}{\hat{\alpha}}}\right)}{d\hat{r}_o}\right|_{\hat{r}_o=1} +\frac{\hat{\eta}}{2\pi}\left(\frac{\lambda^2}{\hat{\alpha}}\right)J_o\left(\sqrt{\hat{\delta}_o^2+\frac{\lambda^2}{\hat{\alpha}}}\right)\right]=0 \end{aligned} \tag{4.292}$$

This is the equation determining the allowable values of $\hat{\delta}_o$ for the reference domain where $-\lambda^2/\hat{\alpha}$ is known. Notice that $\hat{\delta}_o$ depends on R_{out}/R_{in}, $-\lambda^2/\hat{\alpha}$ and on $\hat{\eta}$.

Solving Equation (4.292) determines the values of $\hat{\delta}_o$ for any given values of R_{out}/R_{in}, $-\lambda^2/\hat{\alpha}$ and $\hat{\eta}$. But because $-\lambda^2/\hat{\alpha}$ is held fixed the value of $\hat{\delta}_o$ can be affected by two factors; they are R_{out}/R_{in} and $\hat{\eta}$. The effects of these two factors on the stability of this problem were discussed in full earlier. Therefore, solving Equation (4.292) determines the values of $\hat{\delta}_o$ for any given value of R_{out}/R_{in} and $\hat{\eta}$. For the case where $R_{out}/R_{in}=5$ and $\hat{\eta}=2\pi$ the first value of $\hat{\delta}_o$ is equal to 0.514 which is the one obtained from the previous calculation.

However, the constants A and B are not independent, and one can be found in term of the other. Doing this we write our eigenfunction as

$$\upsilon_o = C\left[-Y_o(R_{out}\sigma_o)J_o(r_o\sigma_o)+Y_o(r_o\sigma_o)J_o(R_{out}\sigma_o)\right] \tag{4.293}$$

where C is an undetermined constant. This equation can be written in fully scaled form, that is

$$\upsilon_o = C\left[-Y_o\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right)J_o(\hat{r}_o\hat{\sigma}_o)+Y_o(\hat{r}_o\hat{\sigma}_o)J_o\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right)\right] \tag{4.294}$$

Our next job is to correct δ_o^2 for the displacement of the pipe. Then δ^2 will be

$$\delta^2 = \delta_o^2 + \varepsilon\delta_1^2 + \frac{1}{2}\varepsilon^2\delta_2^2 + \ldots$$

Now, to find δ_1^2 we must solve Equation (4.274), which is

$$\nabla^2\upsilon_1 + \left(\delta_o^2 + \frac{\lambda^2}{\alpha}\right)\upsilon_1 + \delta_1^2\upsilon_o = 0$$

where

$$\upsilon_1(r_o = R_{out}) = -R_1 \left.\frac{d\upsilon_o}{dr_o}\right|_{R_{out}}$$

and

$$\int_0^{2\pi} \left.\frac{\partial\upsilon_1}{\partial r_o}\right|_{R_{in}} d\theta_o = \eta\left(-\frac{\lambda^2}{\alpha}\right)\upsilon_1\big|_{R_{in}}$$

To obtain δ_1^2 let us multiply Equation (4.274) by υ_o, multiply Equation (4.271) by υ_1 and subtract the second from the first to obtain

$$\nabla\cdot(\upsilon_o\nabla\upsilon_1) - \nabla\cdot(\upsilon_1\nabla\upsilon_o) + \delta_1^2\upsilon_o^2 = 0 \tag{4.295}$$

Now integrate Equation (4.295) over the reference domain and use the divergence theorem to get

$$\begin{aligned} -\int_{R_{in}}^{R_{out}}\int_0^{2\pi}\left(\delta_1^2\upsilon_o^2\right)d\theta_o r_o dr_o &= \left.\int_0^{2\pi} R_{out}\left(\vec{n}\cdot(\upsilon_o\nabla\upsilon_1) - \vec{n}\cdot(\upsilon_1\nabla\upsilon_o)\right)d\theta_o\right|_{r_o=R_{out}} \\ &\quad + \left.\int_0^{2\pi} R_{in}\left(\vec{n}\cdot(\upsilon_o\nabla\upsilon_1) - \vec{n}\cdot(\upsilon_1\nabla\upsilon_o)\right)d\theta_o\right|_{r_o=R_{in}} \end{aligned} \tag{4.296}$$

where $\vec{n}$ is the unit outward normal to the annular region, as shown in Figure 4-8B.

Now the first integral in the right-hand side of Equation (4.296) is zero because $\upsilon_o = 0$ at $r_o = R_{out}$ and because υ_1 at $r_o = R_{out}$, which is given by Equation (4.275), is a multiple of $\cos\theta_o$ and when integrated over $0, 2\pi$ it vanishes. Hence we must pay attention only to the second integral. Because at the rod wall, at $r_o = R_{in}$, υ_o and υ_1 are fixed, then Equation (4.296) reduces to

$$\delta_1^2 \int_{R_{in}}^{R_{out}} \int_0^{2\pi} \upsilon_o^2 d\theta_o r_o dr_o = R_{in}\,\upsilon_o\big|_{r_o=R_{in}} \int_0^{2\pi} \frac{d\upsilon_1}{dr_o}\bigg|_{r_o=R_{in}} d\theta_o - R_{in}\,\upsilon_1\big|_{r_o=R_{in}} \int_0^{2\pi} \frac{d\upsilon_o}{dr_o}\bigg|_{r_o=R_{in}} d\theta_o \quad (4.297)$$

By using Equations (4.273) and (4.276), Equation (4.297) becomes

$$\delta_1^2 \int_{R_{in}}^{R_{out}} \int_0^{2\pi} \upsilon_o^2 d\theta_o r_o dr_o = 0 \quad (4.298)$$

Because υ_o^2 is not zero, then δ_1^2 must be zero which is an expected result and again it happened due to the symmetry of the annulus. And again to learn the effect of a displacement we must go to second order of ε. But there we will need υ_1 which can be determined by solving

$$\nabla^2\upsilon_1 + \left(\delta_o^2 + \frac{\lambda^2}{\alpha}\right)\upsilon_1 = 0 \quad (4.299)$$

which can be written as

$$\nabla^2\upsilon_1 + \sigma_o^2\upsilon_1 = 0 \quad (4.300)$$

where

$$\upsilon_1(r_o = R_{out}) = -R_1 \frac{d\upsilon_o}{dr_o}\bigg|_{R_{out}}$$

and

$$\int_0^{2\pi} \frac{\partial\upsilon_1}{\partial r_o}\bigg|_{R_{in}} d\theta_o = \eta\left(-\frac{\lambda^2}{\alpha}\right)\upsilon_1\big|_{R_{in}}$$

Notice that the υ_1 problem is not an eigenvalue problem since σ_o^2 is known and its general solution is the sum of a homogeneous solution where $-R_1\left(d\upsilon_o/dr_o\right)\Big|_{r_o=R_{out}}$ is set to zero and a particular solution. The homogeneous part is

$$\upsilon_{1_{Homogeneous}} = \dot{C}\left[-Y_o(R_{out}\sigma_o)J_o(r_o\sigma_o)+Y_o(r_o\sigma_o)J_o(R_{out}\sigma_o)\right]=\dot{C}\upsilon_o \tag{4.301}$$

and, because R_1 is a multiple of $\cos\theta_o$, we can get a particular solution in the form

$$\upsilon_{1_{Particular}} = f\left(r_o\right)\cos\theta_o \tag{4.302}$$

Now, by substituting Equation (4.302) into Equation (4.300) and solving for $f\left(r_o\right)$, a particular solution to the υ_1 problem can be written as

$$\upsilon_{1_{Particular}} = \cos\theta_o\left[C_1J_1(r_o\sigma_o)+C_2Y_1(r_o\sigma_o)\right] \tag{4.303}$$

and

$$\upsilon_1 = \dot{C}\upsilon_o + \cos\theta_o\left[C_1J_1(r_o\sigma_o)+C_2Y_1(r_o\sigma_o)\right] \tag{4.304}$$

where the constants C_1 and C_2 can be determined by using Equations (4.275) and (4.276) . Doing this, υ_1 will turn out to be

$$\upsilon_1 = \dot{C}\upsilon_o + \cos\theta_o\left[\frac{Y_1(R_{in}\sigma_o)J_1(r_o\sigma_o)-Y_1(r_o\sigma_o)J_1(R_{in}\sigma_o)}{-Y_1(R_{in}\sigma_o)J_1(R_{out}\sigma_o)+Y_1(R_{out}\sigma_o)J_1(R_{in}\sigma_o)}\right]\frac{d\upsilon_o}{dr_o}\Bigg|_{r_o=R_{out}} \tag{4.305}$$

and $d\upsilon_1/dr_o$, required later, is given by

$$\frac{d\upsilon_1}{dr_o} = \dot{C}\frac{d\upsilon_o}{dr_o} + \cos\theta_o\omega_P\frac{d\upsilon_o}{dr_o}\Bigg|_{r_o=R_{out}} \tag{4.306}$$

where

$$\omega_P = \left[\frac{Y_1(R_{in}\sigma_o)\dfrac{dJ_1(r_o\sigma_o)}{dr_o} - \dfrac{dY_1(r_o\sigma_o)}{dr_o}J_1(R_{in}\sigma_o)}{-Y_1(R_{in}\sigma_o)J_1(R_{out}\sigma_o) + Y_1(R_{out}\sigma_o)J_1(R_{in}\sigma_o)} \right]$$

Notice that these two equations are similar to Equations (4.258) and (4.259) obtained from the previous method. As shown previously, the homogeneous solution of the υ_1 will play no role on the value of δ_2^2 and we drop it from now on.

At order ε^2 the problem is

$$\nabla^2\upsilon_2 + \left(\delta_0^2 + \frac{\lambda^2}{\alpha}\right)\upsilon_2 + \delta_2^2\upsilon_o = 0 \tag{4.307}$$

$$\upsilon_2(r_o = R_{out}) = -2R_1 \left.\frac{d\upsilon_1}{dr_o}\right|_{R_{out}} - R_1^2 \left.\frac{d^2\upsilon_o}{dr_o^2}\right|_{R_{out}} - R_2 \left.\frac{d\upsilon_o}{dr_o}\right|_{R_{out}}$$

and

$$\int_0^{2\pi} \left.\frac{\partial\upsilon_2}{\partial r_o}\right|_{R_{in}} d\theta_o = \eta\left(-\frac{\lambda^2}{\alpha}\right)\upsilon_2\big|_{R_{in}}$$

Now we can get δ_2^2 without determining υ_2 multiply Equation (4.307) by υ_o, multiply Equation (4.271) by υ_2 and subtract the second from the first to obtain

$$\nabla\cdot\left(\upsilon_o\nabla\upsilon_2\right) - \nabla\cdot\left(\upsilon_2\nabla\upsilon_o\right) + \delta_2^2\upsilon_o^2 = 0 \tag{4.308}$$

Now integrate Equation (4.308) over the reference domain and use the divergence theorem to get

$$\begin{aligned} -\int_{R_{in}}^{R_{out}}\int_0^{2\pi}\left(\delta_2^2\upsilon_o^2\right)d\theta_o r_o dr_o &= \int_0^{2\pi} R_{out}\left(\vec{n}\cdot\left(\upsilon_o\nabla\upsilon_2\right) - \vec{n}\cdot\left(\upsilon_2\nabla\upsilon_o\right)\right)d\theta_o\Big|_{r_o=R_{out}} \\ &\quad + \int_0^{2\pi} R_{in}\left(\vec{n}\cdot\left(\upsilon_o\nabla\upsilon_2\right) - \vec{n}\cdot\left(\upsilon_2\nabla\upsilon_o\right)\right)d\theta_o\Big|_{r_o=R_{in}} \end{aligned} \tag{4.309}$$

Then, because υ_o vanishes at $r_o = R_{out}$, which is important as we do not know $\left(d\upsilon_2/dr_o\right)$ at $r_o = R_{out}$, Equation (4.309) becomes

$$\begin{aligned}\delta_2^2 \int_{R_{in}}^{R_{out}} \int_0^{2\pi} \upsilon_o^2 d\theta_o r_o dr_o &= R_{out} \int_0^{2\pi} \upsilon_2\big|_{r_o=R_{out}} \frac{d\upsilon_o}{dr_o}\bigg|_{r_o=R_{out}} d\theta_o \\ &\quad + R_{in} \int_0^{2\pi} \upsilon_o\big|_{r_o=R_{in}} \frac{d\upsilon_2}{dr_o}\bigg|_{r_o=R_{in}} d\theta_o - R_{in} \int_0^{2\pi} \upsilon_2\big|_{r_o=R_{in}} \frac{d\upsilon_o}{dr_o}\bigg|_{r_o=R_{in}} d\theta_o\end{aligned} \tag{4.310}$$

and because $d\upsilon_o/dr_o$ is independent of θ_o and υ_o and υ_1 are fixed at $r_o = R_{in}$, then Equation (4.310) will be

$$\begin{aligned}\delta_2^2 \int_{R_{in}}^{R_{out}} \int_0^{2\pi} \upsilon_o^2 d\theta_o r_o dr_o &= R_{out} \frac{d\upsilon_o}{dr_o}\bigg|_{r_o=R_{out}} \int_0^{2\pi} \left(-2R_1 \frac{d\upsilon_1}{dr_o}\bigg|_{r_o=R_{out}} - R_1^2 \frac{d^2\upsilon_o}{dr_o^2}\bigg|_{r_o=R_{out}} \right. \\ &\quad \left. -R_2 \frac{d\upsilon_o}{dr_o}\bigg|_{r_o=R_{out}} \right) d\theta_o + R_{in}\, \upsilon_o\big|_{r_o=R_{in}} \int_0^{2\pi} \frac{d\upsilon_2}{dr_o}\bigg|_{r_o=R_{in}} d\theta_o - R_{in}\, \upsilon_2\big|_{r_o=R_{in}} \int_0^{2\pi} \frac{d\upsilon_o}{dr_o}\bigg|_{r_o=R_{in}} d\theta_o\end{aligned} \tag{4.311}$$

and by using Equations (4.278), (4.279) and (4.273), Equation (4.311) reduces to

$$\begin{aligned}\delta_2^2 \int_{R_{in}}^{R_{out}} \int_0^{2\pi} \upsilon_o^2 d\theta_o r_o dr_o = R_{out} \frac{d\upsilon_o}{dr_o}\bigg|_{r_o=R_{out}} \int_0^{2\pi} &\left(-2R_1 \frac{d\upsilon_1}{dr_o}\bigg|_{r_o=R_{out}} \right. \\ &\left. -R_1^2 \frac{d^2\upsilon_o}{dr_o^2}\bigg|_{r_o=R_{out}} - R_2 \frac{d\upsilon_o}{dr_o}\bigg|_{r_o=R_{out}} \right) d\theta_o\end{aligned} \tag{4.312}$$

Now, using

$$\frac{d^2\upsilon_o}{dr_o^2}\bigg|_{r_o=R_{out}} = -\frac{1}{R_{out}} \frac{d\upsilon_o}{dr_o}\bigg|_{r_o=R_{out}} \tag{4.313}$$

and Equation (4.306), at $r_o = R_{out}$, without the homogeneous part, viz.,

$$\frac{d\upsilon_1}{dr_o}\bigg|_{r_o=R_{out}} = \cos\theta_o\, \omega_P\big|_{r_o=R_{out}} \frac{d\upsilon_o}{dr_o}\bigg|_{r_o=R_{out}}$$

we get

$$\delta_2^2 \int_{R_{in}}^{R_{out}} \int_0^{2\pi} \upsilon_o^2 d\theta_o r_o dr_o = \left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{out}} \int_0^{2\pi} \left(1 - 2R_{out}\, \omega_P\big|_{r_o=R_{out}} \cos^2\theta_o\right) d\theta_o \tag{4.314}$$

and hence the formula for δ_2^2 reduces to

$$\delta_2^2 = \frac{\left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{out}} \int_0^{2\pi} \left(1 - 2R_{out}\, \omega_P\big|_{r_o=R_{out}} \cos^2\theta_o\right) d\theta_o}{\int_{R_{in}}^{R_{out}} \int_0^{2\pi} \upsilon_o^2 d\theta_o r_o dr_o} \tag{4.315}$$

Notice that the denominator of this equation is always positive.

On evaluating the integrals in the numerator and the denominator we obtain

$$\delta_2^2 = \frac{\left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{out}} \left(1 - R_{out}\, \omega_P\big|_{r_o=R_{out}}\right)}{\left[\frac{R_{out}^2}{2\sigma_o^2}\left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{out}} - \frac{R_{in}^2}{2\sigma_o^2}\left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=R_{in}} - \frac{R_{in}^2}{2}\upsilon_o^2\Big|_{r_o=R_{in}}\right]} \tag{4.316}$$

and in fully scaled form it becomes

$$\hat{\delta}_2^2 = \frac{\left(1 - \hat{\omega}_P\big|_{\hat{r}_o=\frac{R_{out}}{R_{in}}}\right)\left(\frac{d\upsilon_o}{d\hat{r}_o}\right)^2_{\hat{r}_o=\frac{R_{out}}{R_{in}}}}{\left[\left(\frac{R_{out}}{R_{in}}\right)^2 \frac{1}{2\hat{\sigma}_o^2}\left(\frac{d\upsilon_o}{d\hat{r}_o}\right)^2_{\hat{r}_o=\frac{R_{out}}{R_{in}}} - \frac{1}{2\hat{\sigma}_o^2}\left(\frac{d\upsilon_o}{d\hat{r}_o}\right)^2_{\hat{r}_o=1} - \frac{1}{2}\upsilon_o^2\Big|_{\hat{r}_o=1}\right]} \tag{4.317}$$

where

$$\frac{d\upsilon_o}{d\hat{r}_o} = C\left[-Y_o\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right)\frac{dJ_o(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o} + \frac{dY_o(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o}J_o\left(\frac{R_{out}}{R_{in}}\hat{\sigma}_o\right)\right]$$

and

$$\hat{\omega}_P = \frac{R_{out}}{R_{in}} \left[\frac{Y_1(\hat{\sigma}_o)\dfrac{dJ_1(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o} - \dfrac{dY_1(\hat{r}_o\hat{\sigma}_o)}{d\hat{r}_o}J_1(\hat{\sigma}_o)}{-Y_1(\hat{\sigma}_o)J_1(\dfrac{R_{out}}{R_{in}}\hat{\sigma}_o) + Y_1(\dfrac{R_{out}}{R_{in}}\hat{\sigma}_o)J_1(\hat{\sigma}_o)} \right]$$

Notice that δ_2^2 depends on $\hat{r}_o$, whose value can range from one to R_{out}/R_{in}.

Now the sign of δ_2^2 depends only on the sign of the factor $\left(1-\hat{\omega}_P\big|_{\hat{r}_o=R_{out}/R_{in}}\right)$ because the signs of all the other components of Equation (4.317) are known, they are positive. We have been unable to determine the sign of this factor without a calculation. But the calculations show that it is positive and, hence, the sign of δ_2^2 must be positive.

Thus far all the components of, δ^2, are obtained (*i.e.*, δ_o^2, δ_1^2 and δ_2^2). The next step is to determine the value of δ^2 for the distorted annulus, which is defined as

$$\delta^2 = \delta_o^2 + \varepsilon\delta_1^2 + \frac{\varepsilon^2}{2}\delta_2^2 + \ldots$$

and then compare its value with that at the reference state (*i.e.*, δ_o^2). To do this, let us keep in mind that, δ_o^2 and δ_2^2 and ε are positive quantities and δ_1^2 is zero. Then $\hat{\delta}^2$ for the distorted annulus is greater than it is for the reference state. This means that the distorted annulus can tolerate higher heat generation than the symmetric annulus. This is because the effects of the θ-conduction and the heat conduction through the rod overcome the effect of the wider region to the right of the rod.

Based on this we can conclude that, for a perfectly conducting inner wall, a distorted annular region is thermally more stable than a symmetric one. Therefore, the perturbation in this case increases the thermal stability of the annular region. Notice that this result agrees with what we obtained earlier by using the first method (*i.e.*, perturb $-\lambda^2/\alpha$ and hold δ^2 fixed).

Now, the main problem of this chapter (*i.e.*, thermal stability in a distorted annular geometry with heat being produced within the domain), is solved for the three different inner boundary conditions (*i.e.*, isothermal, adiabatic and perfectly conducting inner wall). The calculations are made under the assumption that heat is being transferred only by conduction. The solutions of each of these three different boundary conditions are demonstrated in the sub-sections 4.2.1.1, 4.2.1.2 and 4.2.1.3. Our job now is to show how the boundary conditions changes can affect the thermal stability of this problem. To achieve this, let us compare the results obtained from the three problems solved earlier. And let us begin by comparing the values of $\hat{\sigma}^2$ obtained at the reference states, $\hat{\sigma}_o^2$. To do this, let us plot the values of $\hat{\sigma}_o^2$ obtained from each of the three problems *vs.* R_{out}/R_{in}. This is depicted in Figure 4-8.

In Figure 4-8 we can see that, the curves of the adiabatic and the perfectly conducting inner wall lies exactly on each other[13] (*i.e.*, $\hat{\sigma}_o^2\big|_A = \hat{\sigma}_o^2\big|_P$). And because of this the critical value of $\hat{\delta}$ for the two cases must be the same ($\hat{\delta}^2_{critical}\Big|_A = \hat{\delta}^2_{critical}\Big|_P$). From this we learn that, the thermal stability of a symmetric annulus with adiabatic inner wall is equal to the thermal stability of a symmetric annulus with perfectly conducting inner wall. This result agrees with our earlier work in the planar problem, which is presented and discussed in Chapter 3, where we found that, in the symmetric case, the critical value of δ^2 for the adiabatic and the perfectly conducting central layer are the same. Also in

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[13] The reason for this and its prove were presented and discussed earlier in section 4.2.1.3.

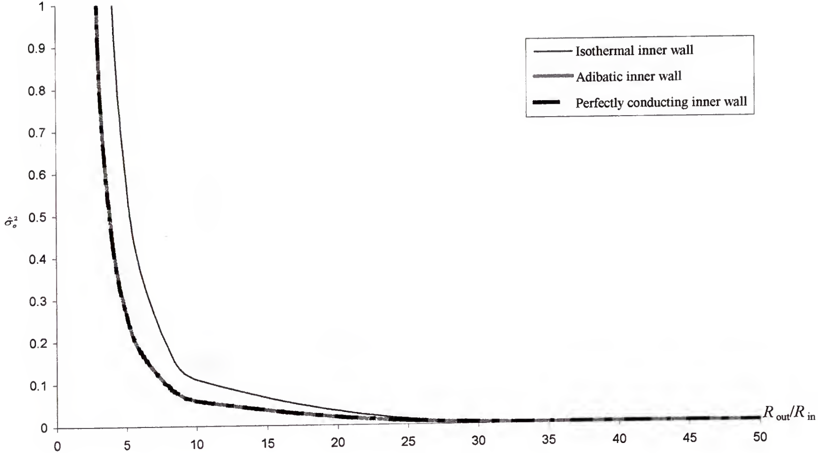

Figure 4-8. Comparison between the values of $\hat{\sigma}_o^2$ obtained at different boundary conditions at the inner wall.

Figure 4-8, we see that the curve of the isothermal inner wall case lies at a higher level than the other two cases. This means that the isothermal inner wall case has bigger values of $\hat{\sigma}_o^2$ than those for the adiabatic and perfectly conducting inner walls (*i.e.*, the isothermal inner wall has the lowest growth rate constant), and, thus, its critical value of $\hat{\delta}^2$ must be the largest. This leads us to conclude that in the reference state the isothermal inner wall case is thermally more stable than the other two cases as it can tolerate more heat generation. This is because in this case both the inner wall and the outer wall serve as heat sinks and then the heat generated can be conducted away from the annular region through the two walls while in the other two cases only the outer wall, which is isothermal, will serve as heat sink. But notice that, $\hat{\sigma}_o^2$ has high values at small values of R_{out}/R_{in} while as the value of R_{out}/R_{in} increases the value of $\hat{\sigma}_o^2$ decreases and the distance between the isothermal curve and the other two curves becomes shorter until, eventually, all the curves lies on each other. This happens because as R_{out}/R_{in} increases the thickness of heat-producing region increases, but we know that thick heat-producing regions are bad conductor and, therefore, they are thermally less stable than thin heat-producing regions.

Now, to see how the displacement affects the thermal stability of these problems, let us compare the results obtained from the distorted annulus. In this case we will begin by comparing the values of the correction factors, $\hat{\sigma}_2^2/\hat{\sigma}_o^2$, then the values of $\hat{\sigma}^2$ obtained from each case. To do this, let us plot the values of $\hat{\sigma}_2^2/\hat{\sigma}_o^2$ obtained from each of the three problems *vs.* R_{out}/R_{in}. This is depicted in Figure 4-9.

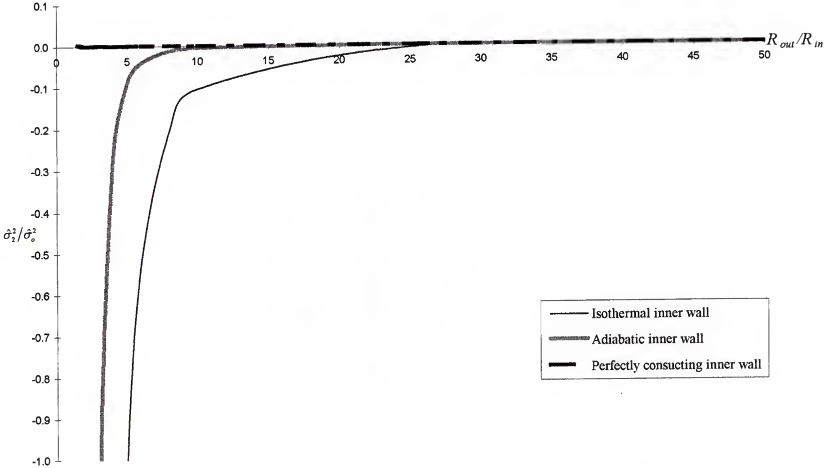

Figure 4-9. Comparison between the values of $\hat{\sigma}_2^2/\hat{\sigma}_o^2$ obtained at different boundary conditions at the inner wall.

From Figure 4-9 we can observe that, the curve of the perfectly conducting inner wall is positive, $\hat{\sigma}_2^2/\hat{\sigma}_o^2 > 0$, but its values are nearly zero. This is because in the distorted annulus, as we explained earlier, the heat generated in the domain will be conducted in both the radial and the azimuthal directions but for this case the heat will also diffuse through the perfectly conducting rod and this diffusion plays an important role on the sign and the magnitude of $\hat{\sigma}_2^2/\hat{\sigma}_o^2$. This is because the rod will act as a short-circuit, where heat can diffuse through towards the thin region on the right-hand side of the rod which improves the thermal stability of the system as it helps the θ-diffusion to overcome the effect of the thick region and because of this the sign of $\hat{\sigma}_2^2/\hat{\sigma}_o^2$ will be positive. On the other hand, this inner rod will act as heat storage, which will store the diffused heat and then release it back into the region as the domain temperature decreases and this back release of the heat reduces the values of $\hat{\sigma}_2^2/\hat{\sigma}_o^2$. This agrees with what we learned from the well-mixed tank problem, which is presented in Appendix E. However, because of the small values of $\hat{\sigma}_2^2/\hat{\sigma}_o^2$, which are almost zero, we can conclude that, in the case of a perfectly conducting inner wall, the rod displacement will have a very small effect on the thermal stability of the annulus and, consequently, we can say that $\hat{\sigma}_o^2\big|_P = \hat{\sigma}^2\big|_P$. This conclusion agrees with our previous work in the planar problem discussed in Chapter 3. That is: the location of the perfectly conducting central layer does not affect the critical value of δ and, consequently, it does not affect the thermal stability of the system.

Moreover, we can see that the curves of the isothermal inner wall and the adiabatic inner wall boundary conditions are located on the negative side of the graph because

$\hat{\sigma}_2^2/\hat{\sigma}_o^2 < 0$. Therefore, for these two cases the displacement will lower the speed of the diffusion and, consequently, the growth constant, $-\lambda^2$, will increase, and because of this the critical value of $\hat{\delta}$ will decrease. Then, these two systems will become thermally less stabile. But notice that the curve of the isothermal inner wall is located at lower level than the adiabatic inner wall curve (*i.e.*, $\hat{\sigma}_2^2/\hat{\sigma}_o^2\big|_I < \hat{\sigma}_2^2/\hat{\sigma}_o^2\big|_A$), which means that thermal diffusion for the case with adiabatic inner wall is higher than it is for the case with isothermal inner wall. This indicates that the adiabatic case is trying to stabilize faster than the isothermal case. To explain this let us recall that in the isothermal case there are two heat sinks, the inner and the outer walls, while in the adiabatic case only the pipe wall will serve as heat sink. Then, the heat generated in the two cases cannot be the same. The heat generated in the adiabatic case must be higher than the heat generated in the isothermal case. And the higher the heat generated the higher the temperature, but the higher the temperature the stronger the temperature gradient and, therefore, the higher the tendency for the heat to get out. Thus, the thermal diffusion in the adiabatic case becomes stronger than it is in the isothermal case. Also notice that the values of $\hat{\sigma}_2^2/\hat{\sigma}_o^2$ are large in magnitudes at small values of R_{out}/R_{in} but as the value of R_{out}/R_{in} increases the value of $\hat{\sigma}_2^2/\hat{\sigma}_o^2$ decreases in magnitude and the difference between the isothermal curve and the adiabatic curve becomes less until, ultimately, the two curves lies on each other. This happens because as R_{out}/R_{in} increases the thickness of heat-producing region increases, and hence thick regions are less sensitive to small perturbation than thin regions.

Now to compare the values of $\hat{\sigma}^2$ let us plot $\hat{\sigma}^2$ obtained from each of the three problems *vs.* R_{out}/R_{in}. This is depicted in Figure 4-10.

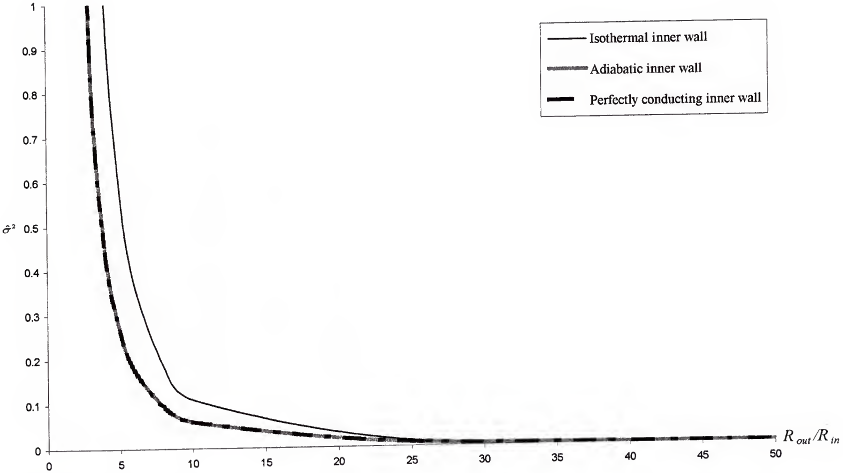

Figure 4-10. Comparison between the values of $\hat{\sigma}^2$ obtained at different boundary conditions with $\varepsilon = 0.5\%$ of R_{out}/R_{in} .

In Figure 4-10 we observe that, the curve of the isothermal inner wall case lies at a higher level than the other two cases. Therefore, the case with isothermal inner wall is thermally more stable than the other two cases even if the displacement decreases its thermal stability and even if its tendency to loss heat is less than that for the adiabatic case, as we learned earlier from Figure 4-9. This is because of the existence of the two heat sinks. Also observe that the curve of the adiabatic inner wall and the curve of the perfectly conducting inner wall lie on each other (*i.e.*, $\hat{\sigma}^2\big|_A = \hat{\sigma}^2\big|_P$). And because of this the critical value of $\hat{\delta}$ for the two cases must be the same (*i.e.*, $\hat{\delta}^2_{critical}\Big|_A = \hat{\delta}^2_{critical}\Big|_P$). From our computation we found that this is always true except for the case with very small R_{out}/R_{in} where the annulus begin to behave like two parallel plane walls. In such a case we found that the values of $\hat{\sigma}^2\big|_A$ becomes less than the values of $\hat{\sigma}^2\big|_P$. This result agrees with our previous work in the planar problem discussed in Chapter 3. That is: when the central layer is displaced, the critical value of δ^2 for the case of perfectly conducting central layer does not change while it decreases for the case of adiabatic central layer.

In the next section, the effect of heat being transferred by both conduction and convection on the thermal stability of a distorted annulus with heat being produced within the domain is studied and solved.

4.2.2 Mathematical Problem in the Case of Heat Transfer by both Conduction and Convection

This section presents a study of the effect of perturbing the inner cylinder on the thermal stability of an annulus in which heat is being produced within the domain (*i.e.*, in the annular region, and carried to the walls by both conduction and convection). In this section we add to one of the problems solved earlier a radial flow outward from the rod to the pipe or inward from the pipe to the rod. To do this, let us consider the problem with isothermal inner wall and let us imagine that the annulus is filled with a porous catalyst that produces heat. To cool this catalyst down we are going to allow fluid to flow either outward or inward by controlling the pressure at the inner and the outer walls. This problem with outward flow is depicted in Figure 4-11. In this simple model the fluid flow can be described via Darcy's law where the velocity of the fluid, $\vec{V}$, is proportional to the pressure gradient, ∇P, viz.,

$$\vec{V} = -\frac{k_p}{\mu}\nabla P$$

where k_p is the permeability of the porous medium and μ is the viscosity of the fluid.

In this case, the case of constant temperature walls, the heat generated in the annular region is removed by convection due to the radial flow and by conduction to the walls, which serve as heat sinks. As depicted in Figure 4-11A, this case has two heat sinks, an inner sink and an outer sink, and, therefore, the maximum temperature is expected to be located somewhere in the middle. Moreover, the thermal stability of the system is expected to increase due to the radial flow. This is because the fluid flows, regardless of its direction, will push the maximum temperature closer to one of the two heat sinks and this will strength the temperature gradient and, consequently, the rate of the heat loss will increase. But when the inner rod is perturbed, as depicted in Figure 4-11B, the factors which control the thermal stability of this problem will increase. They are the wider region to the left of the rod, which reduces the thermal stability as it produces

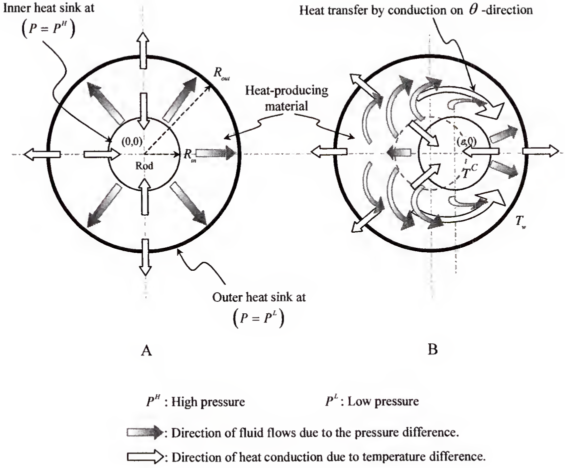
Inner heat sink at
$(P = P^H)$
R_{out}
Heat-producing material
(0,0)
Rod
R_{in}
Outer heat sink at
$(P = P^L)$
A
Heat transfer by conduction on θ -direction
(a,0)
T^C
T_w
B
P^H : High pressure
P^L : Low pressure
: Direction of fluid flows due to the pressure difference.
: Direction of heat conduction due to temperature difference.

Figure 4-11. Heat transfer by conduction and convection between a rod and a pipe both at fixed temperature. A) On-center. B) Off-center.

more heat, the θ -diffusion, which enhances the thermal stability because it can carry heat from the thick region to the thin region and, therefore, to the nearby sinks and the radial flow, which increases the thermal stability as it pushes the maximum temperature closer to one of the two heat sinks. Beside this notice that, in this case the radial flow is not equally spread in the region because the radial pressure gradient is not evenly distributed around the inner cylinder. In this case the radial pressure gradient for the thin region will be higher than it is for the thick region. This is because by shifting the inner cylinder to the right, as depicted in Figure 4-11B, the distance between the rod wall and the pipe wall to the right of the rod will become shorter and since the pressure at the two walls is held fixed then the pressure gradient increases. And on the same measure, we can show that for the thick region the pressure gradient decreases. As a result of this, the radial velocity across the thin region ought to be higher than that for the thick region. Thus, the displacement slows the response of the radial flow in the thick region (*i.e.*, the radial flow in the thick region will not be able to push the maximum temperature close enough to one of the two heat sinks), and this will decrease the thermal stability. Also we know that, in order for the fluid to flow through the region from the rod wall to the pipe wall, as in this case, the pressure at any point in this region must rise above that of the outer wall. And the more distance a point in the region is from the outer wall the higher its pressure must rise. From this we understand that, at any point at a fixed distance from the center of the rod, the pressure in the thick region will be higher than that in the thin region. But we know that the fluid will flow from high-pressure region to low-pressure region. Based on this, the fluid in the thick region will start to flow in the azimuthal direction toward the thin region. Figure 4-11B depicts such a case. Consequently, the radial velocity across the thick region will become weaker and this will slow the response of the radial flow in the thick region, but in the same time the total volumetric flow rate will increase as the azimuthal pressure gradients can come into action and, therefore, the rate of the heat loss will increase and this will enhance the thermal stability of the system. From this we can conclude that the total volumetric flow rate for a distorted annulus is greater than that for a perfectly centered one. This conclusion will be verified during this work.

As usual, we will begin with a simpler problem in order to get some guidance. This simpler problem is the case of parallel plane walls and it is presented in the following.

Heat transfer by conduction and convection from a heat-producing material confined between two, parallel, plane walls. To study this problem, let us consider the simplest planar problem discussed in Chapter 3, *where no central layer exists between the two parallel walls*, and let us assume that the confided space between the two walls is filled with a porous catalyst that produces heat. To cool this catalyst down we must let some fluid flow through the porous medium by controlling the pressure at the two walls. Again, Darcy's law can be used to describe the fluid flow in this model. This problem is depicted in Figure 4-12.

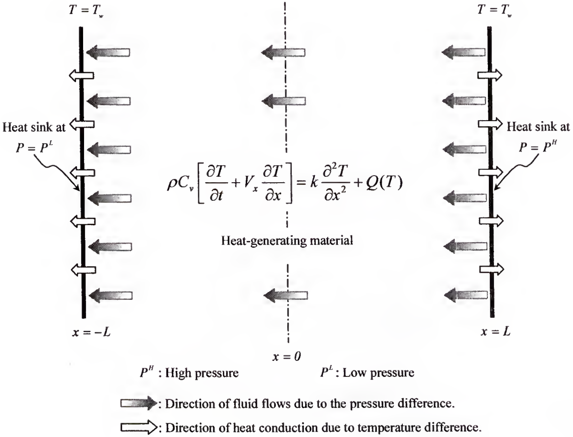
$T = T_w$
$T = T_w$
Heat sink at $P = P^L$
Heat sink at $P = P^H$
$\rho C_v\left[\frac{\partial T}{\partial t} + V_x\frac{\partial T}{\partial x}\right] = k\frac{\partial^2 T}{\partial x^2} + Q(T)$
Heat-generating material
$x = -L$
$x = 0$
$x = L$
P^H : High pressure
P^L : Low pressure
: Direction of fluid flows due to the pressure difference.
: Direction of heat conduction due to temperature difference.

Figure 4-12. Planar geometry when no central layer exists.

Notice that in this problem the two walls are held at constant temperatures (*i.e.*, isothermal walls), then, in the absence of flow, the maximum temperature is expected to be in the center of the region, at $x = 0$. But in the presence of flow the maximum temperature will be either to the left or to the right depending on the direction of the flow and hence, the direction of the flow will not affect the thermal stability of this problem. This is because in either case (*i.e.*, the right to left flow or the left to right flow), the flow will push the maximum temperature closer to one of the two isothermal walls, which serve as heat sinks, and, therefore, the system becomes thermally more stabile. But if one of these two walls, let us say the right-hand wall, is adiabatic then the maximum temperature is expected to be on this wall and, therefore, the direction of the flow will affect the thermal stability. This is because if the fluid flows from right to left then the flow will push the maximum temperature towards the isothermal wall which will enhance the thermal stability. But if the fluid flows from left to right then the flow will bring more heat to the adiabatic wall and then the thermal stability will decrease.

Let us begin by writing the domain equation for this problem. The temperature rise must satisfy the thermal energy equation

$$\rho C_v \left[\frac{\partial T}{\partial t} + V_x \frac{\partial T}{\partial x} \right] = k \frac{\partial^2 T}{\partial x^2} + Q(T) \qquad (4.318)$$

where T is the temperature field, V_x is the velocity of the fluid flows in the x direction and where $Q(T)$ denotes the rate at which heat is produced in the material per unit volume, assumed to be linear function in T. The temperature is subject to two boundary conditions viz.,

$$T(x = -L) = T_w \qquad (4.319)$$

and

$$T(x = L) = T_w \qquad (4.320)$$

Let $Q(T)$ be a linear function of T (*i.e.*, $Q(T) = Q_o + Q_1(T - T_w)$) and set $\dot{T} = T - T_w$. Based on Equation (4.318) $\dot{T}$ must satisfy

$$\rho C_v \left[\frac{\partial \dot{T}}{\partial t} + V_x \frac{\partial \dot{T}}{\partial x} \right] = k \frac{\partial^2 \dot{T}}{\partial x^2} + Q_0 \left(1 + S\dot{T} \right) \tag{4.321}$$

where

$$S = \frac{Q_1}{Q_o}$$

Then rewriting Equation (4.321) in scaled form by defining $\dot{T} = \overline{T}\ T^*$, $V_x = \overline{V}_x\ V_x^*$, $x = \overline{x}\ x^*$ and $t = \overline{t}\ t^*$, where $\overline{T} = 1/S$, $\overline{V}_x = k/\rho C_v L$, $\overline{x} = L$ and $\overline{t} = \rho C_v L^2/k$, we get

$$\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial x^{*2}} - V_x^* \frac{\partial T^*}{\partial x^*} + \delta^2 (1 + T^*)\ , \quad -1 < x^* < 1 \tag{4.322}$$

where

$$\delta^2 = \frac{Q_o S L^2}{k}$$

Again, the value of δ^2 indicates the strength of the heat source and $(1 + T^*)$ indicates how fast it increases as T^* increases.

On dropping the *'s Equation (4.322) will be

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} - V_x \frac{\partial T}{\partial x} + \delta^2 (1 + T)\ , \quad -1 < x < 1 \tag{4.323}$$

and at steady state the temperature rise must satisfy

$$\frac{d^2 T_{steady}}{dx^2} - V_x \frac{dT_{steady}}{dx} + \delta^2 (1 + T_{steady}) = 0 \tag{4.324}$$

where

$$T_{steady}(x = -1) = 0 \tag{4.325}$$

and

$$T_{steady}(x=1)=0 \tag{4.326}$$

From now on we will drop the subscripts *steady* and x from Equation (4.324) and its boundary conditions.

To solve Equation (4.324) we must define its homogeneous and particular solutions. A particular solution can easily be determined viz.,

$$T_p=-1 \tag{4.327}$$

and the homogeneous solution can be found by solving

$$\frac{d^2T}{dx^2}-V\frac{dT}{dx}+\delta^2T=0 \tag{4.328}$$

Trying a solution of the form e^{mx} the values of m must satisfy

$$m^2-Vm+\delta^2=0 \tag{4.329}$$

hence there are two values of m, denoted m_1 and m_2 where

$$m_1=\frac{V+\sqrt{V^2-4\delta^2}}{2} \tag{4.330}$$

and

$$m_2=\frac{V-\sqrt{V^2-4\delta^2}}{2} \tag{4.331}$$

Now our job is to find $\delta_{critical}$ (*i.e.*, δ at which no solution exist), and it appears that three cases present themselves, depending on whether $V^2-4\delta^2$ is positive, zero or negative. These three cases are presented and discussed in the following.

Case I: Set $V^2-4\delta^2>0$, then m_1 and m_2 are real and distinct, and we write the solution to Equation (4.324) in the form

$$T = Ae^{m_1 x} + Be^{m_2 x} - 1 \, , \quad -1 < x < 1 \tag{4.332}$$

To find A and B substitute this formula for T into Equations (4.325) and (4.326) to get

$$Ae^{-m_1} + Be^{-m_2} - 1 = 0 \, , \; x = -1 \tag{4.333}$$

and

$$Ae^{m_1} + Be^{m_2} - 1 = 0 \; , \; x = 1 \tag{4.334}$$

which can be written in matrix form as

$$\begin{bmatrix} e^{-m_1} & e^{-m_2} \\ e^{m_1} & e^{m_2} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.335}$$

To find $\delta_{critical}$ the determinant of the matrix in Equation (4.335) must vanish. But this determinant does not vanish unless $m_1 = m_2$, which cannot be true because in this case the roots must be real and distinct. As a result of this we can find A and B uniquely and then δ cannot be critical. Therefore, this case is rejected because it can give no δ that is critical.

Case II: If $V^2 - 4\delta^2 = 0$, then m_1 and m_2 are equal and the solution to Equation (4.324) is

$$T = Ae^{m_1 x} + Bxe^{m_1 x} - 1 \, , \quad -1 < x < 1 \tag{4.336}$$

Again A and B must be chosen so that T satisfies Equations (4.325) and (4.326) viz.,

$$Ae^{-m_1} - Be^{-m_1} - 1 = 0 \, , \; x = -1 \tag{4.337}$$

and

$$Ae^{m_1} + Be^{m_1} - 1 = 0 \; , \; x = 1 \tag{4.338}$$

or as

$$\begin{bmatrix} e^{-m_1} & -e^{-m_1} \\ e^{m_1} & e^{m_1} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.339}$$

It is clear that this determinant cannot be zero; it is equal to 2. Because of this the values of A and B can be obtained uniquely and, therefore, δ cannot be critical. Again, this case is rejected.

From the prior two cases we can see that, the roots m_1 and m_2 cannot be real. Therefore, they must be complex conjugate numbers.

Case III: If $V^2 - 4\delta^2 < 0$, then m_1 and m_2 are complex conjugates and we write the solution as

$$T = Ae^{mx}\cos(\mathrm{Im}\cdot x) + Be^{mx}\sin(\mathrm{Im}\cdot x) - 1\,, \quad -1 < x < 1 \tag{4.340}$$

where m is the real part of m_1 and Im is its imaginary part viz.,

$$m = \frac{V}{2} \tag{4.341}$$

and

$$\mathrm{Im} = \frac{1}{2}\sqrt{4\delta^2 - V^2} \tag{4.342}$$

To find the constants A and B, turn to the boundary conditions (*i.e.*, Equations (4.325) and (4.326)), whereupon A and B must satisfy

$$Ae^{-\frac{V}{2}}\cos\left(\frac{\sqrt{4\delta^2 - V^2}}{2}\right) - Be^{-\frac{V}{2}}\sin\left(\frac{\sqrt{4\delta^2 - V^2}}{2}\right) - 1 = 0\,, \quad x = -1 \tag{4.343}$$

and

$$Ae^{\frac{V}{2}}\cos\left(\frac{\sqrt{4\delta^2 - V^2}}{2}\right) + Be^{\frac{V}{2}}\sin\left(\frac{\sqrt{4\delta^2 - V^2}}{2}\right) - 1 = 0\,, \quad x = 1 \tag{4.344}$$

which can be written in the form

$$\begin{bmatrix} e^{-\frac{V}{2}}\cos\left(\frac{\sqrt{4\delta^2-V^2}}{2}\right) & -e^{-\frac{V}{2}}\sin\left(\frac{\sqrt{4\delta^2-V^2}}{2}\right) \\ e^{\frac{V}{2}}\cos\left(\frac{\sqrt{4\delta^2-V^2}}{2}\right) & e^{\frac{V}{2}}\sin\left(\frac{\sqrt{4\delta^2-V^2}}{2}\right) \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.345}$$

Now to find the critical value of δ, the determinant of the above matrix must vanish.

Hence $\delta_{critical}$ must satisfy

$$\cos\left(\frac{\sqrt{4\delta^2-V^2}}{2}\right)\sin\left(\frac{\sqrt{4\delta^2-V^2}}{2}\right)+\cos\left(\frac{\sqrt{4\delta^2-V^2}}{2}\right)\sin\left(\frac{\sqrt{4\delta^2-V^2}}{2}\right)=0 \tag{4.346}$$

or

$$\left[\cos\left(\frac{\sqrt{4\delta^2-V^2}}{2}\right)\sin\left(\frac{\sqrt{4\delta^2-V^2}}{2}\right)\right]=0 \tag{4.347}$$

or

$$\sin\left(\sqrt{4\delta^2-V^2}\right)=0 \tag{4.348}$$

whereupon the critical value of δ must be found among

$$\delta^2_{critical}=\frac{V^2}{4}+\frac{n^2\pi^2}{4}\ , \quad n=0,1,2,3,\ldots \tag{4.349}$$

Notice that the lowest $\delta_{critical}$ occurs at $n=0$ but this is not allowed because when $n=0$ we get $V^2-4\delta^2=0$ while in this case (*i.e.*, Case III), we set $V^2-4\delta^2<0$. Hence, the lowest $\delta_{critical}$ will be at $n=1$. Then, at $n=1$

$$\delta^2_{critical}=\frac{V^2}{4}+\frac{\pi^2}{4} \tag{4.350}$$

Now in Chapter 3, where only heat transfer by conduction is considered, we found[14]

$$\delta^2_{critical} = \frac{\pi^2}{4} \tag{4.351}$$

and so we can write

$$\delta^2_{critical}\Big|_{convection} = \delta^2_{critical}\Big|_{conduction} + \frac{V^2}{4} \tag{4.352}$$

hence

$$\delta^2_{critical}\Big|_{convection} > \delta^2_{critical}\Big|_{conduction} \tag{4.353}$$

Then when both conduction and convection heat-transfer modes are considered the critical value of δ increases and, therefore, the system becomes more stable as it can tolerate more heat generation. Thus, our system is stabilized by convection. This result agrees with our earlier expectation.

Now, let us return to the main problem of this section (*i.e.*, the effect of conduction and convection on the thermal stability of a distorted annular region). As we decided earlier we are going to consider the case with *isothermal inner wall.* Notice that without convection this problem was solved in section 4.2.1.1 and now we add in convection.

Let us begin with the right-hand side of Figure 4-11. In this problem the temperature rise can be described by the thermal energy equation

$$\rho C_v \frac{\partial T}{\partial t} = k\nabla^2 T - \rho C_v \vec{V}.\nabla T + Q(T) \tag{4.354}$$

---

[14] Written in completely scaled form.

where T is the temperature field, $\vec{V}$ is the velocity vector and where $Q(T)$ denotes the rate at which heat is produced in the material per unit volume, assumed to be linear function in T. The temperature is subject to two boundary conditions viz.,

$$T\left(r = R_{out}\right) = T_w \tag{4.355}$$

and

$$T\left(r = R(\theta,\varepsilon)\right) = T^C \tag{4.356}$$

and as we mentioned earlier, the fluid flow will be described via Darcy's law viz.,

$$\vec{V} = -\frac{k_p}{\mu}\nabla P \tag{4.357}$$

where ∇P is the pressure gradient, k_p is the permeability of the porous medium and μ is the viscosity of the fluid. The pressure is subject to two boundary conditions viz.,

$$P\left(r = R_{out}\right) = P^L \tag{4.358}$$

and

$$P\left(r = R(\theta,\varepsilon)\right) = P^H \tag{4.359}$$

Before going any further, it is important to mention that for this problem we choose to work with the completely scaled equations from the beginning.

Let $Q(T)$ be a linear function of T (*i.e.*, $Q(T) = Q_o + Q_1(T - T_w)$) and set $\dot{T} = T - T_w$, whence $\dot{T}$ must satisfy

$$\rho C_v \frac{\partial \dot{T}}{\partial t} = k\nabla^2 \dot{T} - \rho C_v \vec{V}.\nabla \dot{T} + Q_o\left(1 + S\dot{T}\right) \tag{4.360}$$

where

$$S = \frac{Q_1}{Q_o}$$

which is the sensitivity of the heat generation to temperature. And on dividing by k we get

$$\frac{\rho C_v}{k}\frac{\partial \dot{T}}{\partial t} = \nabla^2 \dot{T} - \frac{\rho C_v}{k}\vec{V}.\nabla \dot{T} + \frac{Q_o}{k}\left(1 + S\dot{T}\right) \tag{4.361}$$

Then rewrite this equation in scaled form by defining $\dot{T} = \overline{T}\ T^*$, $t = \overline{t}\ t^*$, $r = \overline{r}\ r^*$, $\vec{V} = \overline{V}\ \vec{V}^*$ and $\nabla = \overline{\nabla}\ \nabla^*$ where $\overline{T} = 1/S$, $\overline{t} = \rho C_v R_{in}^2/k$, $\overline{r} = R_{in}$, $\overline{V} = k/\rho C_v R_{in}$ and where $\overline{\nabla} = 1/R_{in}$. Then we get

$$\frac{\partial T^*}{\partial t^*} = \nabla^{*2} T^* - \vec{V}^*.\nabla^* T^* + \delta^2\left(1 + T^*\right) \tag{4.362}$$

where

$$\delta^2 = \frac{Q_o S R_{in}^2}{k}$$

Moreover, the flow equation (*i.e.*, Equation (4.357)), can be written in scaled form by defining $P = \overline{P}\ P^*$ where $\overline{P} = k\mu/k_p \rho C_v$. Then we get

$$\vec{V}^* = -\nabla^* P^* \tag{4.363}$$

On dropping the *'s Equations (4.362) and (4.363) will become

$$\frac{\partial T}{\partial t} = \nabla^2 T - \vec{V}.\nabla T + \delta^2\left(1 + T\right) \tag{4.364}$$

and

$$\vec{V} = -\nabla P \tag{4.365}$$

Then, by substituting Equation (4.365) into Equation (4.364) we get

$$\frac{\partial T}{\partial t} = \nabla^2 T + \nabla P.\nabla T + \delta^2\left(1 + T\right) \tag{4.366}$$

whence the steady state temperature rise must satisfy

$$0 = \nabla^2 T_{steady} + \nabla P.\nabla T_{steady} + \delta^2 \left(1 + T_{steady}\right) \tag{4.367}$$

The input variables of this problem are ∇P, ε and δ^2. All the input variables are held fixed except δ^2. The only output variable, at steady state, is therefore T_{steady}. It depends on r and θ.

Based on what we learned earlier in section 4.1 and section 4.2.1 and in Chapters 2 and 3, this problem can be solved by the domain perturbation method. But since we want to determine the critical value of δ^2 which is defined when $T \to \infty$, the domain perturbation method cannot be applied directly to the domain equations. Instead we must introduce the corresponding eigenvalue problem and solve it for a reference case, the symmetric annular region (*i.e.*, Figure 4-11A). The eigenvalue problem of the distorted annulus geometry is then solved by perturbing the equations about the reference state.

Then, by following the steps given in section 4.1, the eigenvalue problem is found to be

$$\nabla^2 \upsilon + \nabla P.\nabla \upsilon + \sigma^2 \upsilon = 0 \tag{4.368}$$

where $\sigma^2 = \delta^2 + \lambda^2$,

$$\upsilon\left(r = R_{out}\right) = 0 \tag{4.369}$$

and

$$\upsilon\left(r = R\left(\theta, \varepsilon\right)\right) = 0 \tag{4.370}$$

However, similar to what we learned in the previous sections, anything that lowers $-\lambda^2$ makes the system more stable (*i.e.*, allows greater values of δ^2), and what lowers $-\lambda^2$ is σ^2, the measure of the diffusion strength, and since the critical value of δ^2 is reached when $-\lambda^2 = 0$, then larger values of δ^2 corresponds to larger values of σ^2 which means higher thermal-diffusion rates. Thus, the effect of perturbing the inner cylinder on the thermal stability of the annulus can be seen from the change in the value of σ^2.

Now, for this problem (*i.e.*, Figure 4-11B), the pipe wall presents no problem; it is the rod wall whose surface is displaced. For such a case, the surface of the rod is given by $r = R(\theta, \varepsilon)$ where R is expanded as

$$R = R_o + \varepsilon R_1 + \frac{\varepsilon^2}{2} R_2 + \ldots \qquad (4.371)$$

where

$$R_o = 1, \quad R_1 = \cos\theta_o \text{ and } R_2 = -\sin^2\theta_o$$

Again, the solution of the reference case will be denoted by affixing a subscript zero to its variables while the subscripts one and two will denote the factors in the expansion to first and second order of ε, respectively.

The expansion of $\vec{n}$, see Appendix D, can be written as

$$\vec{n} = \vec{n}_o + \varepsilon \vec{n}_1 + \frac{\varepsilon^2}{2} \vec{n}_2 + \ldots \qquad (4.372)$$

where

$$\vec{n}_o = \vec{i}_r, \quad \vec{n}_1 = \vec{i}_\theta \sin\theta_o \text{ and } \vec{n}_2 = -\vec{i}_r \frac{\sin^2\theta_o}{2}$$

The expansions of υ and P on the domain, by using the rule given in Appendix A, can be written as

$$\upsilon = \upsilon_o + \varepsilon \upsilon_1 + \frac{\varepsilon^2}{2} \upsilon_2 + \ldots \qquad (4.373)$$

and

$$P = P_o + \varepsilon P_1 + \frac{\varepsilon^2}{2} P_2 + \ldots \tag{4.374}$$

and the expansions of υ and P at $r = R(\theta,\varepsilon)$, are given by

$$\upsilon(R,\theta) = \upsilon_o + \varepsilon\left(\upsilon_1 + R_1 \frac{d\upsilon_o}{dr_o}\right) + \frac{\varepsilon^2}{2}\left(\upsilon_2 + 2R_1 \frac{d\upsilon_1}{dr_o} + R_1^2 \frac{d^2\upsilon_o}{dr_o^2} + R_2 \frac{d\upsilon_o}{dr_o}\right) + \ldots \tag{4.375}$$

and

$$P(R,\theta) = P_o + \varepsilon\left(P_1 + R_1 \frac{dP_o}{dr_o}\right) + \frac{\varepsilon^2}{2}\left(P_2 + 2R_1 \frac{dP_1}{dr_o} + R_1^2 \frac{d^2 P_o}{dr_o^2} + R_2 \frac{dP_o}{dr_o}\right) + \ldots \tag{4.376}$$

The expansion of $\vec{V}$ is

$$\vec{V} = \vec{V}_o + \varepsilon \vec{V}_1 + \frac{\varepsilon^2}{2} \vec{V}_2 + \ldots \tag{4.377}$$

The expansion of the eigenvalue, σ^2, will be

$$\sigma^2 = \sigma_o^2 + \varepsilon \sigma_1^2 + \frac{\varepsilon^2}{2} \sigma_2^2 + \ldots \tag{4.378}$$

which is same as

$$\sigma^2 = \left(\delta^2 + \lambda^2\right) = \left(\delta_o^2 + \lambda_o^2\right) + \varepsilon\left(\delta_1^2 + \lambda_1^2\right) + \frac{\varepsilon^2}{2}\left(\delta_2^2 + \lambda_2^2\right) + \ldots \tag{4.379}$$

The equations to be solved with their boundary conditions[15] are: First at order ε^o

$$\nabla^2 \upsilon_o + \nabla P_o . \nabla \upsilon_o + \sigma_o^2 \upsilon_o = 0 \tag{4.380}$$

where

$$\upsilon_o(r_o = R_{out}) = 0 \tag{4.381}$$

and

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[15] Note that the perturbed boundary conditions are no longer isothermal because they try to simulate isothermal by putting a condition other than isothermal on the reference domain.

$$\upsilon_o(r_o = 1) = 0 \tag{4.382}$$

For order ε^1

$$\nabla^2\upsilon_1 + \nabla P_o.\nabla\upsilon_1 + \nabla P_1.\nabla\upsilon_o + \sigma_o^2\upsilon_1 + \sigma_1^2\upsilon_o = 0 \tag{4.383}$$

where

$$\upsilon_1(r_o = R_{out}) = 0 \tag{4.384}$$

and

$$\upsilon_1(r_o = 1) = -R_1 \left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=1} \tag{4.385}$$

For order ε^2

$$\frac{1}{2}\nabla^2\upsilon_2 + \frac{1}{2}\nabla P_o.\nabla\upsilon_2 + \nabla P_1.\nabla\upsilon_1 + \frac{1}{2}\nabla P_2.\nabla\upsilon_o + \frac{1}{2}\sigma_o^2\upsilon_2 + \sigma_1^2\upsilon_1 + \frac{1}{2}\sigma_2^2\upsilon_o = 0 \tag{4.386}$$

where

$$\upsilon_2(r_o = R_{out}) = 0 \tag{4.387}$$

and

$$\upsilon_2(r_o = 1) = -2R_1\left.\frac{d\upsilon_1}{dr_o}\right|_{r_o=1} - R_1^2\left.\frac{d^2\upsilon_o}{dr_o^2}\right|_{r_o=1} - R_2\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=1} \tag{4.388}$$

To explain where Equations (4.385) and (4.388) come from, notice that the expansion of υ at $r = R(\theta,\varepsilon)$, is given by Equation (4.375), namely by

$$\upsilon(R,\theta) = \upsilon_o + \varepsilon\left(\upsilon_1 + R_1\frac{d\upsilon_o}{dr_o}\right) + \frac{\varepsilon^2}{2}\left(\upsilon_2 + 2R_1\frac{d\upsilon_1}{dr_o} + R_1^2\frac{d^2\upsilon_o}{dr_o^2} + R_2\frac{d\upsilon_o}{dr_o}\right) + \dots$$

and that $\upsilon(R,\theta)$ must vanish for all values of ε.

Now, in order to solve these problems we must first solve for $\nabla P_o, \nabla P_1$ and ∇P_2. This calculation is presented in Appendix F. From this calculation we obtain

$$P_o = P^H - \frac{\left(P^H - P^L\right)}{\ln\left(R_{out}\right)}\ln\left(r_o\right) \tag{4.389}$$

Moreover,

$$\frac{dP_o}{dr_o} = -\frac{\left(P^H - P^L\right)}{\ln\left(R_{out}\right)}\frac{1}{r_o} \tag{4.390}$$

and

$$\frac{d^2P_o}{dr_o^2} = \frac{\left(P^H - P^L\right)}{\ln\left(R_{out}\right)}\frac{1}{r_o^2} \tag{4.391}$$

These derivatives will be useful later on. However, by setting $C = \left(P^H - P^L\right)/\ln\left(R_{out}\right)$ Equations (4.390) and (4.391) can be rewritten as

$$\frac{dP_o}{dr_o} = -\frac{\left(P^H - P^L\right)}{\ln\left(R_{out}\right)}\frac{1}{r_o} = -\frac{C}{r_o} \tag{4.392}$$

and

$$\frac{d^2P_o}{dr_o^2} = \frac{\left(P^H - P^L\right)}{\ln\left(R_{out}\right)}\frac{1}{r_o^2} = \frac{C}{r_o^2} \tag{4.393}$$

where the magnitude of C and its sign determine the strength of the flow and its direction, respectively. In this problem, positive values of C will stand for outward flow (*i.e.*, from the rod to the pipe), while negative values of C will stand for inward flow (*i.e.*, from the pipe to the rod). Also we obtain

$$P_1 = \cos\theta_o \frac{1}{\left(R_{out}^2 - 1\right)}\left(r_o - \frac{R_{out}^2}{r_o}\right)\left.\frac{dP_o}{dr_o}\right|_{r_o=1} \tag{4.394}$$

and by this, there obtains

$$\frac{dP_1}{dr_o} = \cos\theta_o \frac{1}{\left(R_{out}^2 - 1\right)}\left(1 + \frac{R_{out}^2}{r_o^2}\right)\frac{dP_o}{dr_o}\bigg|_{r_o=1} \tag{4.395}$$

and

$$\frac{dP_1}{d\theta_o} = -\sin\theta_o \frac{1}{\left(R_{out}^2 - 1\right)}\left(r_o - \frac{R_{out}^2}{r_o}\right)\frac{dP_o}{dr_o}\bigg|_{r_o=1} \tag{4.396}$$

P_2 is found to be

$$P_2 = -\left[\frac{2}{\left(R_{out}^2 - 1\right)\ln\left(R_{out}\right)}\ln\left(\frac{R_{out}}{r_o}\right) + \frac{1}{\left(R_{out}^2 - 1\right)^2}\left(\frac{R_{out}^4}{r_o^2} - r_o^2\right)\cos 2\theta_o\right]\frac{dP_o}{dr_o}\bigg|_{r_o=1} \tag{4.397}$$

and by this, there obtains

$$\frac{dP_2}{dr_o} = \left[\frac{2}{\left(R_{out}^2 - 1\right)\ln\left(R_{out}\right)r_o} + \frac{2}{\left(R_{out}^2 - 1\right)^2}\left(\frac{R_{out}^4}{r_o^3} + r_o\right)\cos 2\theta_o\right]\frac{dP_o}{dr_o}\bigg|_{r_o=1} \tag{4.398}$$

and

$$\frac{dP_2}{d\theta_o} = \left[\frac{2}{\left(R_{out}^2 - 1\right)^2}\left(\frac{R_{out}^4}{r_o^2} - r_o^2\right)\sin 2\theta_o\right]\frac{dP_o}{dr_o}\bigg|_{r_o=1} \tag{4.399}$$

These derivatives will be useful later on.

Now, given that P_o, P_1 and P_2 are obtained then the energy problems (*i.e.*, Equations (4.380), (4.383) and (4.386)), can be solved but before doing that let us try to verify our previous conclusion on the total volumetric flow rate for this problem that is: the total volumetric flow rate for a distorted annulus is greater than that for a perfectly centered one. To achieve this, we must determine the velocity at each order of ε (*i.e.*, $\vec{V}_o, \vec{V}_1$ and $\vec{V}_2$), which can be done by perturbing Equation (4.365). From this we find that for order ε^o, which is the reference case

$$\vec{V}_o = \vec{i}_r V_{r_o} = -\vec{i}_r \frac{dP_o}{dr_o} = -\nabla P_o \tag{4.400}$$

and by using Equation (4.390) we get

$$V_{r_o} = -\frac{dP_o}{dr_o} = \frac{\left(P^H - P^L\right)}{\ln\left(R_{out}\right)}\frac{1}{r_o} \tag{4.401}$$

For order ε^1,

$$\vec{V}_1 = \vec{i}_r V_{r_1} + \vec{i}_\theta V_{\theta_1} = -\left(\vec{i}_r \frac{\partial P_1}{\partial r_o} + \frac{\vec{i}_\theta}{r_o}\frac{\partial P_1}{\partial \theta_o}\right) = -\nabla P_1 \tag{4.402}$$

and from Equations (4.395) and (4.396) we find

$$V_{r_1} = -\frac{\partial P_1}{\partial r_o} = -\cos\theta_o \frac{1}{\left(R_{out}^2 - 1\right)}\left(1 + \frac{R_{out}^2}{r_o^2}\right)\left.\frac{dP_o}{dr_o}\right|_{r_o=1} \tag{4.403}$$

and

$$V_{\theta_1} = -\frac{1}{r_o}\frac{\partial P_1}{\partial \theta_o} = \sin\theta_o \frac{1}{\left(R_{out}^2 - 1\right)}\left(r_o - \frac{R_{out}^2}{r_o}\right)\left.\frac{dP_o}{dr_o}\right|_{r_o=1} \tag{4.404}$$

For order ε^2,

$$\vec{V}_2 = \vec{i}_r V_{r_2} + \vec{i}_\theta V_{\theta_2} = -\left(\vec{i}_r \frac{\partial P_2}{\partial r_o} + \frac{\vec{i}_\theta}{r_o}\frac{\partial P_2}{\partial \theta_o}\right) = -\nabla P_2 \tag{4.405}$$

and on using Equations (4.398) and (4.399) we obtain

$$V_{r_2} = -\frac{\partial P_2}{\partial r_o} = -\left[\frac{2}{\left(R_{out}^2 - 1\right)\ln\left(R_{out}\right)r_o} + \frac{2}{\left(R_{out}^2 - 1\right)^2}\left(\frac{R_{out}^4}{r_o^3} + r_o\right)\cos 2\theta_o\right]\left.\frac{dP_o}{dr_o}\right|_{r_o=1} \tag{4.406}$$

and

$$V_{\theta_2} = -\frac{1}{r_o}\frac{\partial P_2}{\partial \theta_o} = -\left[\frac{2}{\left(R_{out}^2 - 1\right)^2}\left(\frac{R_{out}^4}{r_o^3} - r_o\right)\sin 2\theta_o\right]\left.\frac{dP_o}{dr_o}\right|_{r_o=1} \tag{4.407}$$

Now, the total volumetric flow rate, Q_f, per unit length across the boundaries, which is given by

$$Q_f = \int dA\vec{n} \cdot \vec{V} \tag{4.408}$$

can be obtained. It is important to notice that $V_\theta = 0$ at the boundary. Hence, because of radial symmetry Q_f becomes

$$Q_f = R_{out} \int_o^{2\pi} V_r d\theta \tag{4.409}$$

which can be written as

$$Q_f = R_{out} \int_o^{2\pi} \left(V_{r_o} + \varepsilon V_{r_1} + \frac{\varepsilon^2}{2} V_{r_2} + \ldots \right) d\theta_o \tag{4.410}$$

or

$$Q_f = Q_{f_o} + \varepsilon Q_{f_1} + \frac{\varepsilon^2}{2} Q_{f_2} + \ldots \tag{4.411}$$

where Q_{f_o} is the total volumetric flow rate for the symmetric case and where Q_{f_1} and Q_{f_2} are the corrections terms because of the displacement of the inner cylinder. From this the total volumetric flow rate for the reference case, Q_{f_o}, is found to be

$$Q_{f_o} = R_{out} \int_o^{2\pi} V_{r_o} \Big|_{r_o = R_{out}} d\theta_o = 2\pi \frac{\left(P^H - P^L\right)}{\ln\left(R_{out}\right)} > 0 \tag{4.412}$$

Q_{f_1} which is the coefficient of the first perturbation term (*i.e.*, ε^1), found as

$$Q_{f_1} = R_{out} \int_o^{2\pi} V_{r_1} \Big|_{r_o = R_{out}} d\theta_o = -\frac{2}{\left(R_{out}^2 - 1\right)} \frac{dP_o}{dr_o}\Bigg|_{r_o = 1} \int_o^{2\pi} \cos\theta_o d\theta_o = 0 \tag{4.413}$$

whence

$$Q_{f_1} = 0 \tag{4.414}$$

This means that Q_{f_1} which appears in Equation (4.411) has no effect on the total Q_f. This is an expected result due to the symmetry of the annulus. Hence, to see the effect of the displacement of the inner rod on the volumetric flow rate we must go to the second order of ε.

The value of Q_{f_2} which is the coefficient of the second perturbation term (*i.e.*, ε^2), is equal to

$$Q_{f_2} = R_{out} \int_{o}^{2\pi} V_{r_2}\Big|_{r_o = R_{out}} d\theta_o = -\int_{o}^{2\pi} \left[\frac{2}{\left(R_{out}^2 - 1\right) \ln\left(R_{out}\right)} + \frac{4R_{out}^2}{\left(R_{out}^2 - 1\right)^2} \cos 2\theta_o \right] \frac{dP_o}{dr_o}\bigg|_{r_o = 1} d\theta_o \tag{4.415}$$

whereupon integrating this equation we obtain

$$Q_{f_2} = \left[\frac{4\pi\left(P^H - P^L\right)}{\left(R_{out}^2 - 1\right)\left(\ln\left(R_{out}\right)\right)^2} \right] > 0 \tag{4.416}$$

Observe that the value of Q_{f_2} is always positive because R_{out} can only be greater than one. In addition to this, notice that this calculation of Q_f is similar to the calculation of the total heat-transfer rate done earlier in Chapter 2.

However, since ε, Q_{f_o} and Q_{f_2} are positive quantities and Q_{f_1} is zero, we can observe that the total volumetric flow rate, Q_f, for the distorted annulus is greater than it is for the perfectly centered one. This result agrees with what we expected from the picture argument.

Now let us solve the energy problems at each order of ε (*i.e.*, Equations (4.380), (4.383) and (4.386)).

Begin with the problem at order ε^o (*i.e.*, Equation (4.380)), namely

$$\nabla^2 \upsilon_o + \nabla P_o . \nabla \upsilon_o + \sigma_o^2 \upsilon_o = 0$$

where

$$\upsilon_o(r_o = R_{out}) = 0$$

$$\upsilon_o(r_o = 1) = 0$$

and

$$\nabla P_o \cdot \nabla \upsilon_o = \vec{i}_r \frac{\partial P_o}{\partial r_o} \cdot \left(\vec{i}_r \frac{\partial \upsilon_o}{\partial r_o} \right) = \frac{\partial P_o}{\partial r_o} \frac{\partial \upsilon_o}{\partial r_o} \qquad (4.417)$$

because in the reference state (*i.e.*, the symmetric case), there is no variation of P or υ in the θ-direction. Then, by using Equation (4.392) we get

$$\nabla P_o \cdot \nabla \upsilon_o = -\frac{C}{r_o} \frac{\partial \upsilon_o}{\partial r_o} \qquad (4.418)$$

and hence Equation (4.380) becomes

$$\frac{\partial^2 \upsilon_o}{\partial r_o^2} + \frac{(1-C)}{r_o} \frac{\partial \upsilon_o}{\partial r_o} + \frac{1}{r_o^2} \frac{\partial^2 \upsilon_o}{\partial \theta_o^2} + \sigma_o^2 \upsilon_o = 0 \qquad (4.419)$$

So long as υ_o does not depend on θ_o, this problem has solutions of the form

$$\mathcal{R}(r_o) \cos M\theta_o \qquad (4.420)$$

and

$$\mathcal{R}(r_o) \sin M\theta_o \qquad (4.421)$$

where M must be one of 0,1,2,… in order to satisfy periodicity and where $\mathcal{R}$ must satisfy

$$r_o^2 \frac{d^2 \mathcal{R}}{dr_o^2} + r_o (1-C) \frac{d\mathcal{R}}{dr_o} + \left(r_o^2 \sigma_o^2 - M^2 \right) \mathcal{R} = 0 \qquad (4.422)$$

where $\mathcal{R}$ is subject to two boundary conditions viz.,

$$\mathcal{R}(r_o = R_{out}) = 0 \tag{4.423}$$

and

$$\mathcal{R}(r_o = 1) = 0 \tag{4.424}$$

To solve Equation (4.422) we can view it as a special case of the generalized form of the Bessel's equation[16], which means

$$\begin{aligned} x^2 \frac{d^2 y}{dx^2} + x\left(a + 2bx^{\hat{r}}\right)\frac{dy}{dx} \\ + \left[c + dx^{2s} - b\left(1 - a - \hat{r}\right)x^{\hat{r}} + b^2 x^{2\hat{r}}\right]y = 0 \end{aligned} \tag{4.425}$$

whose solution is given by

$$y = x^{(1-a)/2} e^{-\left(bx^{\hat{r}}/\hat{r}\right)} \left[c_1 Z_p\left(\frac{\sqrt{|d|}}{s} x^s\right) + c_2 Z_{-p}\left(\frac{\sqrt{|d|}}{s} x^s\right) \right] \tag{4.426}$$

where

$$p = \frac{1}{s}\sqrt{\left(\frac{1-a}{2}\right)^2 - c} \tag{4.427}$$

and where Z_p denotes a Bessel function whose kind depends on the value of $\sqrt{d}/s$. If $\sqrt{d}/s$ is real and p is not zero or an integer, Z_p denotes J_p, Z_{-p} denotes J_{-p}; if p is zero or an integer, Z_p denotes J_p, Z_{-p} denotes Y_p. If $\sqrt{d}/s$ is imaginary and p is not zero or an integer, Z_p denotes I_p, Z_{-p} denotes I_{-p}; if p is zero or an integer, Z_p

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[16] A clear explanation on how to use this generalized form is given in Mickley, H.S., Sherwood, T.K. and Reed, C.E., 1957, Applied Mathematics in Chemical Engineering, McGraw-Hill, 2nd Ed., New York.

denotes I_p, Z_{-p} denotes K_p, where I_p and K_p are the modified Bessel functions of order p of the first and second kind, respectively.

A comparison of Equation (4.422) and Equation (4.425) reveals that, when $\hat{r} = 1$, $b = 0$ and $s = 1$ the generalized form reduces to

$$x^2 \frac{d^2 y}{dx^2} + ax\frac{dy}{dx} + \left[c + dx^2\right] y = 0 \tag{4.428}$$

and hence the two equations match if $a = (1 - C)$, $c = -M^2$ and $d = \sigma_o^2$. Let us focus on the case $M = 0$. Then

$$p = \sqrt{\left(\frac{C}{2}\right)^2} = \frac{C}{2} \tag{4.429}$$

and we begin with $C > 0$, corresponding to outflow. Then, to write a solution for this problem let us pick the case where p is zero or integer. Based on this, C must be an even integer. Therefore, the solution of Equation (4.422) will be

$$\mathcal{R} = r_o^{\left(\frac{C}{2}\right)} \left[AJ_{\frac{C}{2}}(\sigma_o r_o) + BY_{\frac{C}{2}}(\sigma_o r_o) \right] \tag{4.430}$$

Therefore, the eigenfunctions will take the form

$$r_o^{\left(\frac{C}{2}\right)} \left[AJ_{\frac{C}{2}}(\sigma_o r_o) + BY_{\frac{C}{2}}(\sigma_o r_o) \right] \cos M\theta_o \tag{4.431}$$

and

$$r_o^{\left(\frac{C}{2}\right)} \left[AJ_{\frac{C}{2}}(\sigma_o r_o) + BY_{\frac{C}{2}}(\sigma_o r_o) \right] \sin M\theta_o \tag{4.432}$$

But as υ_o is independent of θ_o, Equation (4.419) reduces to

$$\frac{\partial^2 \upsilon_o}{\partial r_o^2} + \frac{(1 - C)}{r_o} \frac{\partial \upsilon_o}{\partial r_o} + \sigma_o^2 \upsilon_o = 0 \tag{4.433}$$

and the eigenfunction takes the form

$$\upsilon_o = r_o^{\left(\frac{C}{2}\right)}\left[AJ_{\frac{C}{2}}(\sigma_o r_o) + BY_{\frac{C}{2}}(\sigma_o r_o)\right] \tag{4.434}$$

where A, B and σ_o must be determined. To do this, apply the boundary conditions given by Equations (4.381) and (4.382) to get

$$AJ_{\frac{C}{2}}(R_{out}\sigma_o) + BY_{\frac{C}{2}}(R_{out}\sigma_o) = 0 \tag{4.435}$$

and

$$AJ_{\frac{C}{2}}(\sigma_o) + BY_{\frac{C}{2}}(\sigma_o) = 0 \tag{4.436}$$

Equations (4.435) and (4.436) can be written in matrix form as

$$\begin{bmatrix} J_{\frac{C}{2}}(\sigma_o R_{out}) & Y_{\frac{C}{2}}(\sigma_o R_{out}) \\ J_{\frac{C}{2}}(\sigma_o) & Y_{\frac{C}{2}}(\sigma_o) \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.437}$$

To find a solution other than $A = 0$ and $B = 0$ the determinant of the this matrix must vanish, which means

$$J_{\frac{C}{2}}(\sigma_o R_{out})Y_{\frac{C}{2}}(\sigma_o) - J_{\frac{C}{2}}(\sigma_o)Y_{\frac{C}{2}}(\sigma_o R_{out}) = 0 \tag{4.438}$$

and the roots of this equation determine the values of σ_o for any given value of R_{out} and C. Notice that σ_o^2 depends only on R_{out} and C.

To determine the values of σ_o using Equation (4.438), let us first hold R_{out} fixed and play with C to see how the strength of the radial flow and, therefore, the strength of the convection heat-transfer mode affects the thermal stability. This calculation is presented in Table 4-6 for $C > 0$, outward flow, and for $C < 0$, inward flow. Then, let us hold C fixed and change R_{out} to see how the thickness of the region affects the results. This calculation is presented in Table 4-7.

Table 4-6. Values of σ_o at $R_{out} = 5$ and various values of C's.

$C > 0$	σ_o	$C < 0$	σ_o
0	0.763	0	0.763
2	0.847	-2	0.847
4	1.044	-4	1.044

Table 4-7. Values of σ_o at $C = 2$ and various values of R_{out}.

R_{out}	σ_o
5	0.847
10	0.394
15	0.259

Now, from Table 4-6 we observe that at $C = 0$ (*i.e.*, when there is no flow), we recover the conduction problem of constant temperature walls and that the value of σ_o is just that given in Table 4-1 for the same value of R_{out}. Then observe that for any given value of C the value of σ_o is the same regardless of the sign of C. This is due to the symmetry of the annulus. Because at any given value of r_o, $1 < r_o < R_{out}$, the ratio of the total heat generated to the total heat lost must always be the same. From this we can conclude that in the symmetric case the values of σ_o are independent of the direction of the flow and, therefore, the thermal stability of this problem is independent of the direction of the flow.

Again the critical value of δ^2 corresponds to $-\lambda^2 = 0$ and hence $\delta^2_{critical}$ is given by σ^2 and one can note from Table 4-6 that in the symmetric case the radial flow, regardless of its direction, will increase the thermal stability of this system. And from table 4-7 one can notice that thick heat-producing regions are thermally less stable than thin heat-producing regions.

However, the constants A and B which appear in Equation (4.434) are not independent and then one can be found in term of the other. Doing this we can write

$$\upsilon_o = \widehat{B} r_o^{\left(\frac{C}{2}\right)} \left[-Y_{\frac{C}{2}}(\sigma_o) J_{\frac{C}{2}}(\sigma_o r_o) + Y_{\frac{C}{2}}(\sigma_o r_o) J_{\frac{C}{2}}(\sigma_o) \right] \tag{4.439}$$

where $\widehat{B}$ is a constant that can have any value except zero. From this we can get

$$\begin{aligned} \frac{d\upsilon_o}{dr_o} = \widehat{B} \Bigg(& \frac{C}{2} r_o^{\left(\frac{C}{2}-1\right)} \left[-Y_{\frac{C}{2}}(\sigma_o) J_{\frac{C}{2}}(\sigma_o r_o) + Y_{\frac{C}{2}}(\sigma_o r_o) J_{\frac{C}{2}}(\sigma_o) \right] \\ & + r_o^{\left(\frac{C}{2}\right)} \left[-Y_{\frac{C}{2}}(\sigma_o) \frac{dJ_{\frac{C}{2}}(\sigma_o r_o)}{dr_o} + \frac{dY_{\frac{C}{2}}(\sigma_o r_o)}{dr_o} J_{\frac{C}{2}}(\sigma_o) \right] \Bigg) \end{aligned} \tag{4.440}$$

and therefore

$$\left. \frac{d\upsilon_o}{dr_o} \right|_{r_o=1} = \widehat{B} \left[-Y_{\frac{C}{2}}(\sigma_o) \left. \frac{dJ_{\frac{C}{2}}(\sigma_o r_o)}{dr_o} \right|_{r_o=1} + \left. \frac{dY_{\frac{C}{2}}(\sigma_o r_o)}{dr_o} \right|_{r_o=1} J_{\frac{C}{2}}(\sigma_o) \right] \tag{4.441}$$

This derivative will be useful later on.

Now we have the eigenfunctions in the reference state and the corresponding eigenvalues. These must be corrected as the rod is moved off-center and so we turn to the calculation of σ_1^2 and σ_2^2 which will allows us to estimate the effect of the displacement on σ^2 via

$$\sigma^2 = \sigma_o^2 + \varepsilon \sigma_1^2 + \frac{\varepsilon^2}{2} \sigma_2^2 + \dots$$

Again, we focus our attention to the most dangerous case, $M = 0$, and determine the corrections to the eigenvalues corresponding to the axisymmetry eigenfunctions.

Now, to find σ_1^2 we must solve Equation (4.383), which is

$$\nabla^2 \upsilon_1 + \nabla P_o . \nabla \upsilon_1 + \nabla P_1 . \nabla \upsilon_o + \sigma_o^2 \upsilon_1 + \sigma_1^2 \upsilon_o = 0$$

where

$$\upsilon_1 (r_o = R_{out}) = 0$$

and

$$\upsilon_1 (r_o = 1) = -R_1 \left. \frac{d\upsilon_o}{dr_o} \right|_{r_o = 1}$$

and where υ_1 is driven by $d\upsilon_o / dr_o$ and where

$$\nabla P_o \cdot \nabla \upsilon_1 = \vec{i}_r \frac{\partial P_o}{\partial r_o} \cdot \left(\vec{i}_r \frac{\partial \upsilon_1}{\partial r_o} + \frac{\vec{i}_\theta}{r_o} \frac{\partial \upsilon_1}{\partial \theta_o} \right) = \frac{\partial P_o}{\partial r_o} \frac{\partial \upsilon_1}{\partial r_o} \tag{4.442}$$

and

$$\nabla P_1 \cdot \nabla \upsilon_o = \left(\vec{i}_r \frac{\partial P_1}{\partial r_o} + \frac{\vec{i}_\theta}{r_o} \frac{\partial P_1}{\partial \theta_o} \right) \cdot \left(\vec{i}_r \frac{\partial \upsilon_o}{\partial r_o} \right) = \frac{\partial P_1}{\partial r_o} \frac{d\upsilon_o}{dr_o} \tag{4.443}$$

Then by using Equations (4.392) and (4.395) we can write these two equations in the following form

$$\nabla P_o \cdot \nabla \upsilon_1 = -\frac{C}{r_o} \frac{\partial \upsilon_1}{\partial r_o} \tag{4.444}$$

and

$$\nabla P_1 \cdot \nabla \upsilon_o = -Cb(r_o) \frac{d\upsilon_o}{dr_o} \cos\theta_o \tag{4.445}$$

where

$$b(r_o) = \frac{1}{\left(R_{out}^2 - 1 \right)} \left(1 + \frac{R_{out}^2}{r_o^2} \right) \tag{4.446}$$

Hence, Equation (4.383) becomes

$$\frac{\partial^2 \upsilon_1}{\partial r_o^2}+\frac{(1-C)}{r_o}\frac{\partial \upsilon_1}{\partial r_o}+\frac{1}{r_o^2}\frac{\partial^2 \upsilon_1}{\partial \theta_o^2}+\sigma_o^2\upsilon_1+\sigma_1^2\upsilon_o-Cb(r_o)\frac{d\upsilon_o}{dr_o}\cos\theta_o=0 \tag{4.447}$$

where the general solution of υ_1 can take the following form

$$\upsilon_1=f_o+f_1\cos\theta_o \tag{4.448}$$

Then, by substituting Equation (4.448) into Equation (4.447) and into its boundary conditions (*i.e.*, Equations (4.384) and (4.385)), we get two ordinary differential equations one in f_o and one in f_1, they are

$$\frac{d^2 f_o}{dr_o^2}+\frac{(1-C)}{r_o}\frac{df_o}{dr_o}+\sigma_o^2 f_o+\sigma_1^2\upsilon_o=0 \tag{4.449}$$

where f_o must satisfy

$$f_o(r_o=R_{out})=0 \tag{4.450}$$

and

$$f_o(r_o=1)=0 \tag{4.451}$$

and

$$\frac{d^2 f_1}{dr_o^2}+\frac{(1-C)}{r_o}\frac{df_1}{dr_o}+\left(\sigma_o^2-\frac{1}{r_o^2}\right)f_1-Cb(r_o)\frac{d\upsilon_o}{dr_o}=0 \tag{4.452}$$

where f_1 must satisfy

$$f_1(r_o=R_{out})=0 \tag{4.453}$$

and

$$f_1(r_o=1)=-\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=1} \tag{4.454}$$

Notice that σ_1^2 appears only in the f_o equation (*i.e.*, Equation (4.449)), therefore to determine σ_1^2 we must use Equation (4.449). To begin, let us rewrite Equations (4.449) and (4.433) in self-adjoint form.[17] To do this, multiply Equation (4.449) by the function $r_o^{(1-C)}$, which gives

$$r_o^{(1-C)}\frac{d^2 f_o}{dr_o^2}+(1-C)r_o^{(-C)}\frac{df_o}{dr_o}+\sigma_o^2 r_o^{(1-C)}f_o+\sigma_1^2 r_o^{(1-C)}\upsilon_o=0 \tag{4.455}$$

which can be written as

$$\frac{d}{dr_o}\left(r_o^{(1-C)}\frac{df_o}{dr_o}\right)+\sigma_o^2 r_o^{(1-C)}f_o+\sigma_1^2 r_o^{(1-C)}\upsilon_o=0 \tag{4.456}$$

and multiply Equation (4.433) by the function $r_o^{(1-C)}$, to get

$$r_o^{(1-C)}\frac{d^2\upsilon_o}{dr_o^2}+(1-C)r_o^{(-C)}\frac{d\upsilon_o}{dr_o}+\sigma_o^2 r_o^{(1-C)}\upsilon_o=0 \tag{4.457}$$

which is just

$$\frac{d}{dr_o}\left(r_o^{(1-C)}\frac{d\upsilon_o}{dr_o}\right)+\sigma_o^2 r_o^{(1-C)}\upsilon_o=0 \tag{4.458}$$

Then, multiply Equation (4.456) by υ_o, which gives

$$\upsilon_o\frac{d}{dr_o}\left(r_o^{(1-C)}\frac{df_o}{dr_o}\right)+\sigma_o^2 r_o^{(1-C)}\upsilon_o f_o+\sigma_1^2 r_o^{(1-C)}\upsilon_o^2=0 \tag{4.459}$$

and multiply Equation (4.458) by f_o, to get

$$f_o\frac{d}{dr_o}\left(r_o^{(1-C)}\frac{d\upsilon_o}{dr_o}\right)+\sigma_o^2 r_o^{(1-C)}f_o\upsilon_o=0 \tag{4.460}$$

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[17] The method to do this is explained in Weinberger, H.F., 1995, A First Course in Partial Differential Equations, Dover Publications, Inc., New York.

Then, subtract Equation (4.460) from Equation (4.459), to obtain

$$\upsilon_o \frac{d}{dr_o}\left(r_o^{(1-C)} \frac{df_o}{dr_o} \right) - f_o \frac{d}{dr_o}\left(r_o^{(1-C)} \frac{d\upsilon_o}{dr_o} \right) = -\sigma_1^2 r_o^{(1-C)} \upsilon_o^2 \tag{4.461}$$

Now integrate Equation (4.461) over the reference domain, which gives

$$\int_1^{R_{out}} \upsilon_o \frac{d}{dr_o}\left(r_o^{(1-C)} \frac{df_o}{dr_o} \right) dr_o - \int_1^{R_{out}} f_o \frac{d}{dr_o}\left(r_o^{(1-C)} \frac{d\upsilon_o}{dr_o} \right) dr_o = -\sigma_1^2 \int_1^{R_{out}} r_o^{(1-C)} \upsilon_o^2 dr_o \tag{4.462}$$

Then, by applying the integration by part on the left-hand side of Equation (4.462) we obtain

$$r_o^{(1-C)} \upsilon_o \frac{df_o}{dr_o}\bigg|_1^{R_{out}} - r_o^{(1-C)} f_o \frac{d\upsilon_o}{dr_o}\bigg|_1^{R_{out}} = -\sigma_1^2 \int_1^{R_{out}} r_o^{(1-C)} \upsilon_o^2 dr_o \tag{4.463}$$

Now the left-hand side of Equation (4.463) is zero because of $\upsilon_o(r_o = R_{out}) = 0$, $\upsilon_o(r_o = 1) = 0$, $f_o(r_o = R_{out}) = 0$ and $f_o(r_o = 1) = 0$, hence Equation (4.463) reduces to

$$\sigma_1^2 \int_1^{R_{out}} r_o^{(1-C)} \upsilon_o^2 dr_o = 0 \tag{4.464}$$

Because υ_o^2 is not zero, σ_1^2 must be zero and there is no effect on the stability of our problem at order ε. This is an expected result due to the symmetry of the annulus. Hence to learn the effect of a displacement we must go to second order of ε. But there we will need υ_1 and we must go on and determine υ_1 by setting $\sigma_1^2 = 0$. Thus, the υ_1 problem will reduce to

$$\frac{\partial^2 \upsilon_1}{\partial r_o^2} + \frac{(1-C)}{r_o}\frac{\partial \upsilon_1}{\partial r_o} + \frac{1}{r_o^2}\frac{\partial^2 \upsilon_1}{\partial \theta_o^2} + \sigma_o^2 \upsilon_1 - Cb(r_o)\frac{d\upsilon_o}{dr_o}\cos\theta_o = 0 \tag{4.465}$$

which is not an eigenvalue problem since σ_o^2 is known. Based on this, the f_o problem (*i.e.*, Equation (4.449)), reduces to

$$\frac{d^2 f_o}{dr_o^2}+\frac{(1-C)}{r_o}\frac{df_o}{dr_o}+\sigma_o^2 f_o=0 \tag{4.466}$$

where f_o must satisfy

$$f_o\left(r_o=R_{out}\right)=0$$

and

$$f_o\left(r_o=1\right)=0$$

The solution to this problem is

$$f_o=\dot{B}r_o^{\left(\frac{C}{2}\right)}\left[-Y_{\frac{C}{2}}(\sigma_o)J_{\frac{C}{2}}(\sigma_o r_o)+Y_{\frac{C}{2}}(\sigma_o r_o)J_{\frac{C}{2}}(\sigma_o)\right] \tag{4.467}$$

where $\dot{B}$ is an undetermined constant that can have any value except zero.

The next step now is to find f_1 by solving Equation (4.452) which can be written as

$$\frac{d^2 f_1}{dr_o^2}+\frac{(1-C)}{r_o}\frac{df_1}{dr_o}+\left(\sigma_o^2-\frac{1}{r_o^2}\right)f_1=Cb\left(r_o\right)\frac{d\upsilon_o}{dr_o} \tag{4.468}$$

where f_1 must satisfy

$$f_1\left(r_o=R_{out}\right)=0$$

and

$$f_1\left(r_o=1\right)=-\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=1}$$

Now, by defining $f_1=\hat{f}_1+\hat{\hat{f}}_1$ the f_1 problem can be divided into two simpler problems, they are

$$\frac{d^2\hat{f}_1}{dr_o^2}+\frac{(1-C)}{r_o}\frac{d\hat{f}_1}{dr_o}+\left(\sigma_o^2-\frac{1}{r_o^2}\right)\hat{f}_1=0 \tag{4.469}$$

where $\hat{f}_1$ must satisfy

$$\hat{f}_1\left(r_o=R_{out}\right)=0 \tag{4.470}$$

and

$$\hat{f}_1\left(r_o=1\right)=-\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=1} \tag{4.471}$$

and

$$\frac{d^2\hat{\hat{f}}_1}{dr_o^2}+\frac{(1-C)}{r_o}\frac{d\hat{\hat{f}}_1}{dr_o}+\left(\sigma_o^2-\frac{1}{r_o^2}\right)\hat{\hat{f}}_1=Cb\left(r_o\right)\frac{d\upsilon_o}{dr_o} \tag{4.472}$$

where $\hat{\hat{f}}_1$ must satisfy

$$\hat{\hat{f}}_1\left(r_o=R_{out}\right)=0 \tag{4.473}$$

and

$$\hat{\hat{f}}_1\left(r_o=1\right)=0 \tag{4.474}$$

The solution of these two problem (*i.e.*, Equations (4.469) and (4.472)), can be obtained by the same method used to solve the υ_o problem. Based on this, a solution to the $\hat{f}_1$ problem can be written as

$$\hat{f}_1=r_o^{\left(\frac{C}{2}\right)}\left[C_1J_p(\sigma_o r_o)+C_2J_{-p}(\sigma_o r_o)\right] \tag{4.475}$$

where

$$p=\sqrt{\frac{C^2+4}{4}} \tag{4.476}$$

which cannot be zero or integer because we have decided to set C to be an even integer, and where the constants C_1 and C_2 can be determined by using Equations (4.470) and (4.471). Doing this, $\hat{f}_1$ will turn out to be

$$\hat{f}_1 = r_o^{\left(\frac{C}{2}\right)}\left[\frac{J_{-p}(\sigma_o R_{out})J_p(\sigma_o r_o) - J_p(\sigma_o R_{out})J_{-p}(\sigma_o r_o)}{-J_{-p}(\sigma_o R_{out})J_p(\sigma_o) + J_p(\sigma_o R_{out})J_{-p}(\sigma_o)}\right]\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=1} \tag{4.477}$$

and therefore

$$\begin{aligned}\frac{d\hat{f}_1}{dr_o} = \Bigg(&\frac{C}{2} r_o^{\left(\frac{C}{2}-1\right)}\left[\frac{J_{-p}(\sigma_o R_{out})J_p(\sigma_o r_o) - J_p(\sigma_o R_{out})J_{-p}(\sigma_o r_o)}{-J_{-p}(\sigma_o R_{out})J_p(\sigma_o) + J_p(\sigma_o R_{out})J_{-p}(\sigma_o)}\right] + \\ &r_o^{\left(\frac{C}{2}\right)}\left[\frac{J_{-p}(\sigma_o R_{out})\dfrac{dJ_p(\sigma_o r_o)}{dr_o} - J_p(\sigma_o R_{out})\dfrac{dJ_{-p}(\sigma_o r_o)}{dr_o}}{-J_{-p}(\sigma_o R_{out})J_p(\sigma_o) + J_p(\sigma_o R_{out})J_{-p}(\sigma_o)}\right]\Bigg)\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=1}\end{aligned} \tag{4.478}$$

and

$$\left.\frac{d\hat{f}_1}{dr_o}\right|_{r_o=1} = \left(-\frac{C}{2} + \left[\frac{J_{-p}(\sigma_o R_{out})\left.\dfrac{dJ_p(\sigma_o r_o)}{dr_o}\right|_{r_o=1} - J_p(\sigma_o R_{out})\left.\dfrac{dJ_{-p}(\sigma_o r_o)}{dr_o}\right|_{r_o=1}}{-J_{-p}(\sigma_o R_{out})J_p(\sigma_o) + J_p(\sigma_o R_{out})J_{-p}(\sigma_o)}\right]\right)\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=1} \tag{4.479}$$

These derivatives will become useful later on.

In the case of the $\hat{\hat{f}}_1$ problem (*i.e.*, Equation (4.472)), notice that the general solution is the sum of a homogeneous solution, $\hat{\hat{f}}_{1_{Homogeneous}}$, where $Cb(r_o)(d\upsilon_o/dr_o)$ is set to zero and a particular solution, $\hat{\hat{f}}_{1_{Particular}}$. The homogeneous part is

$$\hat{\hat{f}}_{1_{Homogeneous}} = r_o^{\left(\frac{C}{2}\right)}\left[e_1 J_p(\sigma_o r_o) + e_2 J_{-p}(\sigma_o r_o)\right] \tag{4.480}$$

and then the particular solution will be

$$\hat{\hat{f}}_{1_{Particular}} = r_o^{\left(\frac{C}{2}\right)}\left[h_1 J_p(\sigma_o r_o) + h_2 J_{-p}(\sigma_o r_o)\right] \tag{4.481}$$

where

$$h_1 = -\int_1^r \frac{Cb(r_o)\frac{d\upsilon_o}{dr_o} r_o^{\left(\frac{C}{2}\right)} J_{-p}(\sigma_o r_o)}{w} dr_o \tag{4.482}$$

and

$$h_2 = \int_1^r \frac{Cb(r_o)\frac{d\upsilon_o}{dr_o} r_o^{\left(\frac{C}{2}\right)} J_{p}(\sigma_o r_o)}{w} dr_o \tag{4.483}$$

and where w is the Wronskian of $J_p(\sigma_o r_o)$ and $J_{-p}(\sigma_o r_o)$, which means

$$w = r_o^C\left[J_p(\sigma_o r_o)\frac{dJ_{-p}(\sigma_o r_o)}{dr_o} - J_{-p}(\sigma_o r_o)\frac{dJ_p(\sigma_o r_o)}{dr_o}\right] \tag{4.484}$$

whence

$$\begin{aligned} w' = \frac{dw}{dr_o} = Cr_o^{(C-1)}\left[J_p(\sigma_o r_o)\frac{dJ_{-p}(\sigma_o r_o)}{dr_o} - J_{-p}(\sigma_o r_o)\frac{dJ_p(\sigma_o r_o)}{dr_o}\right] + \\ r_o^C\left[J_p(\sigma_o r_o)\frac{d^2 J_{-p}(\sigma_o r_o)}{dr_o^2} - J_{-p}(\sigma_o r_o)\frac{d^2 J_p(\sigma_o r_o)}{dr_o^2}\right] \end{aligned} \tag{4.485}$$

which will be useful later on.

The Wronskian, w, of this problem can be written as

$$w = \frac{a}{r_o^{(1-C)}} \tag{4.486}$$

where a is a constant that can have any value except zero. To prove Equation (4.486) is true, let us consider the $\hat{\hat{f}}_{1_{Homogeneous}}$ problem, which is

$$\frac{d^2\hat{\hat{f}}_{1_{Homogeneous}}}{dr_o^2}+\frac{(1-C)}{r_o}\frac{d\hat{\hat{f}}_{1_{Homogeneous}}}{dr_o}+\left(\sigma_o^2-\frac{1}{r_o^2}\right)\hat{\hat{f}}_{1_{Homogeneous}}=0 \qquad (4.487)$$

where the homogeneous solution is given by Equation (4.480). Then, write Equation (4.487) in a self-adjoin form by multiplying it by $r_o^{(1-C)}$, to get

$$r_o^{(1-C)}\frac{d^2\hat{\hat{f}}_{1_{Homogeneous}}}{dr_o^2}+r_o^{(1-C)}\frac{(1-C)}{r_o}\frac{d\hat{\hat{f}}_{1_{Homogeneous}}}{dr_o}+\left(\sigma_o^2-\frac{1}{r_o^2}\right)r_o^{(1-C)}\hat{\hat{f}}_{1_{Homogeneous}}=0 \quad (4.488)$$

Now set $\psi=r_o^{(1-C)}$, therefore, $\psi'=(d\psi/dr_o)=(1-C)r_o^{-C}$ and hence Equation (4.488) becomes

$$\psi\frac{d^2\hat{\hat{f}}_{1_{Homogeneous}}}{dr_o^2}+\psi'\frac{d\hat{\hat{f}}_{1_{Homogeneous}}}{dr_o}+\left(\sigma_o^2-\frac{1}{r_o^2}\right)\psi\hat{\hat{f}}_{1_{Homogeneous}}=0 \qquad (4.489)$$

Next, substitute the homogeneous solution into this equation. To do this, substitute $\hat{\hat{f}}_{1_{Homogeneous}}=r_o^{\left(\frac{C}{2}\right)}J_p(\sigma_o r_o)$ into Equation (4.489), to get

$$\psi\frac{d^2\left(r_o^{\left(\frac{C}{2}\right)}J_p(\sigma_o r_o)\right)}{dr_o^2}+\psi'\frac{d\left(r_o^{\left(\frac{C}{2}\right)}J_p(\sigma_o r_o)\right)}{dr_o}+\left(\sigma_o^2-\frac{1}{r_o^2}\right)\psi r_o^{\left(\frac{C}{2}\right)}J_p(\sigma_o r_o)=0 \qquad (4.490)$$

and substitute $\hat{\hat{f}}_{1_{Homogeneous}}=r_o^{\left(\frac{C}{2}\right)}J_{-p}(\sigma_o r_o)$ into Equation (4.489), to get

$$\psi\frac{d^2\left(r_o^{\left(\frac{C}{2}\right)}J_{-p}(\sigma_o r_o)\right)}{dr_o^2}+\psi'\frac{d\left(r_o^{\left(\frac{C}{2}\right)}J_{-p}(\sigma_o r_o)\right)}{dr_o}+\left(\sigma_o^2-\frac{1}{r_o^2}\right)\psi r_o^{\left(\frac{C}{2}\right)}J_{-p}(\sigma_o r_o)=0 \qquad (4.491)$$

Then, multiply Equation (4.490) by $r_o^{\left(\frac{C}{2}\right)}J_{-p}(\sigma_o r_o)$, to obtain

$$\psi r_o^{\left(\frac{C}{2}\right)} J_{-p}(\sigma_o r_o) \frac{d^2\left(r_o^{\left(\frac{C}{2}\right)} J_p(\sigma_o r_o)\right)}{dr_o^2} + \psi' r_o^{\left(\frac{C}{2}\right)} J_{-p}(\sigma_o r_o) \frac{d\left(r_o^{\left(\frac{C}{2}\right)} J_p(\sigma_o r_o)\right)}{dr_o} + \left(\sigma_o^2 - \frac{1}{r_o^2}\right) \psi r_o^C J_{-p}(\sigma_o r_o) J_p(\sigma_o r_o) = 0 \tag{4.492}$$

and multiply Equation (4.491) by $r_o^{\left(\frac{C}{2}\right)} J_p(\sigma_o r_o)$, which will become

$$\psi r_o^{\left(\frac{C}{2}\right)} J_p(\sigma_o r_o) \frac{d^2\left(r_o^{\left(\frac{C}{2}\right)} J_{-p}(\sigma_o r_o)\right)}{dr_o^2} + \psi' r_o^{\left(\frac{C}{2}\right)} J_p(\sigma_o r_o) \frac{d\left(r_o^{\left(\frac{C}{2}\right)} J_{-p}(\sigma_o r_o)\right)}{dr_o} + \left(\sigma_o^2 - \frac{1}{r_o^2}\right) \psi r_o^C J_p(\sigma_o r_o) J_{-p}(\sigma_o r_o) = 0 \tag{4.493}$$

Then, subtract Equation (4.493) from Equation (4.492), to get

$$\psi r_o^{\left(\frac{C}{2}\right)} \left(J_{-p}(\sigma_o r_o) \frac{d^2\left(r_o^{\left(\frac{C}{2}\right)} J_p(\sigma_o r_o)\right)}{dr_o^2} - J_p(\sigma_o r_o) \frac{d^2\left(r_o^{\left(\frac{C}{2}\right)} J_{-p}(\sigma_o r_o)\right)}{dr_o^2} \right) = -\psi' r_o^{\left(\frac{C}{2}\right)} \left(J_{-p}(\sigma_o r_o) \frac{d\left(r_o^{\left(\frac{C}{2}\right)} J_p(\sigma_o r_o)\right)}{dr_o} - J_p(\sigma_o r_o) \frac{d\left(r_o^{\left(\frac{C}{2}\right)} J_{-p}(\sigma_o r_o)\right)}{dr_o} \right) \tag{4.494}$$

where

$$\frac{d^2\left(r_o^{\left(\frac{C}{2}\right)} J_p(\sigma_o r_o)\right)}{dr_o^2} = \frac{C}{2}\left(\frac{C}{2} - 1\right) r_o^{\left(\frac{C}{2}-2\right)} J_p(\sigma_o r_o) + C r_o^{\left(\frac{C}{2}-1\right)} \frac{dJ_p(\sigma_o r_o)}{dr_o} + r_o^{\left(\frac{C}{2}\right)} \frac{d^2 J_p(\sigma_o r_o)}{dr_o^2} \tag{4.495}$$

$$\frac{d^2\left(r_o^{\left(\frac{C}{2}\right)}J_{-p}(\sigma_o r_o)\right)}{dr_o^2}=\frac{C}{2}\left(\frac{C}{2}-1\right)r_o^{\left(\frac{C}{2}-2\right)}J_{-p}(\sigma_o r_o)+Cr_o^{\left(\frac{C}{2}-1\right)}\frac{dJ_{-p}(\sigma_o r_o)}{dr_o}+r_o^{\left(\frac{C}{2}\right)}\frac{d^2J_{-p}(\sigma_o r_o)}{dr_o^2} \tag{4.496}$$

$$\frac{d\left(r_o^{\left(\frac{C}{2}\right)}J_p(\sigma_o r_o)\right)}{dr_o}=\frac{C}{2}r_o^{\left(\frac{C}{2}-1\right)}J_p(\sigma_o r_o)+r_o^{\left(\frac{C}{2}\right)}\frac{dJ_p(\sigma_o r_o)}{dr_o} \tag{4.497}$$

and

$$\frac{d\left(r_o^{\left(\frac{C}{2}\right)}J_{-p}(\sigma_o r_o)\right)}{dr_o}=\frac{C}{2}r_o^{\left(\frac{C}{2}-1\right)}J_{-p}(\sigma_o r_o)+r_o^{\left(\frac{C}{2}\right)}\frac{dJ_{-p}(\sigma_o r_o)}{dr_o} \tag{4.498}$$

Thus, Equation (4.494) becomes

$$\begin{aligned}\psi\Bigg(&Cr_o^{(C-1)}\left[J_{-p}(\sigma_o r_o)\frac{dJ_p(\sigma_o r_o)}{dr_o}-J_p(\sigma_o r_o)\frac{dJ_{-p}(\sigma_o r_o)}{dr_o}\right]\\&+r_o^{C}\left[J_{-p}(\sigma_o r_o)\frac{d^2J_p(\sigma_o r_o)}{dr_o^2}-J_p(\sigma_o r_o)\frac{d^2J_{-p}(\sigma_o r_o)}{dr_o^2}\right]\Bigg)=\\&-\psi' r_o^{C}\left(J_{-p}(\sigma_o r_o)\frac{d\left(J_p(\sigma_o r_o)\right)}{dr_o}-J_p(\sigma_o r_o)\frac{d\left(J_{-p}(\sigma_o r_o)\right)}{dr_o}\right)\end{aligned} \tag{4.499}$$

From Equations (4.484) and (4.485) we can observe that Equation (4.499) can be written as

$$\psi w'=-\psi' w \tag{4.500}$$

or

$$\frac{w'}{w}=-\frac{\psi'}{\psi} \tag{4.501}$$

which is just

$$\left(\ln(w)\right)' = -\left(\ln(\psi)\right)' \tag{4.502}$$

by integrating Equation (4.502) we get

$$\ln(w) = -\ln(\psi) + a \tag{4.503}$$

Then, take the exponential of this equation, which is

$$e^{\ln(w)} = e^{-\ln(\psi)+a} \tag{4.504}$$

and from this we obtain

$$w = \frac{a}{\psi} = \frac{a}{r_o^{(1-C)}} \tag{4.505}$$

Hence, Equations (4.482) and (4.483) will become

$$h_1 = -\frac{C}{a}\int_1^r b(r_o)\frac{d\upsilon_o}{dr_o} r_o^{\left(1-\frac{C}{2}\right)} J_{-p}(\sigma_o r_o)dr_o \tag{4.506}$$

and

$$h_2 = \frac{C}{a}\int_1^r b(r_o)\frac{d\upsilon_o}{dr_o} r_o^{\left(1-\frac{C}{2}\right)} J_p(\sigma_o r_o)dr_o \tag{4.507}$$

where

$$a = r_o\left[J_p(\sigma_o r_o)\frac{dJ_{-p}(\sigma_o r_o)}{dr_o} - J_{-p}(\sigma_o r_o)\frac{dJ_p(\sigma_o r_o)}{dr_o}\right]$$

Therefore, the solution to the $\hat{\hat{f}}_1$ problem (Equation (4.472)), will turn out to be

$$\begin{aligned}\hat{\hat{f}}_1 = r_o^{\left(\frac{C}{2}\right)}\Bigg[& e_1 J_p(\sigma_o r_o) + e_2 J_{-p}(\sigma_o r_o) \\ & -\frac{C}{a} J_p(\sigma_o r_o)\int_1^r b(r_o)\frac{d\upsilon_o}{dr_o} r_o^{\left(1-\frac{C}{2}\right)} J_{-p}(\sigma_o r_o)dr_o \\ & +\frac{C}{a} J_{-p}(\sigma_o r_o)\int_1^r b(r_o)\frac{d\upsilon_o}{dr_o} r_o^{\left(1-\frac{C}{2}\right)} J_p(\sigma_o r_o)dr_o\Bigg]\end{aligned} \tag{4.508}$$

where the constants e_1 and e_2 can be determined by using Equations (4.473) and (4.474). Doing this, $\hat{\hat{f}}_1$ will turn out to be

$$\begin{aligned}
\hat{\hat{f}}_1 = r_o^{\left(\frac{C}{2}\right)} \Bigg[& \frac{-J_{-p}(\sigma_o)J_p(\sigma_o r_o)+J_p(\sigma_o)J_{-p}(\sigma_o r_o)}{-J_{-p}(\sigma_o)J_p(\sigma_o R_{out})+J_p(\sigma_o)J_{-p}(\sigma_o R_{out})} \times \\
& \Bigg\{ \frac{C}{a} J_p(\sigma_o R_{out}) \int_1^{R_{out}} b(r_o) \frac{d\upsilon_o}{dr_o} r_o^{\left(1-\frac{C}{2}\right)} J_{-p}(\sigma_o r_o)\, dr_o \\
& \quad - \frac{C}{a} J_{-p}(\sigma_o R_{out}) \int_1^{R_{out}} b(r_o) \frac{d\upsilon_o}{dr_o} r_o^{\left(1-\frac{C}{2}\right)} J_p(\sigma_o r_o)\, dr_o \Bigg\} \\
& - \frac{C}{a} J_p(\sigma_o r_o) \int_1^{r} b(r_o) \frac{d\upsilon_o}{dr_o} r_o^{\left(1-\frac{C}{2}\right)} J_{-p}(\sigma_o r_o)\, dr_o \\
& + \frac{C}{a} J_{-p}(\sigma_o r_o) \int_1^{r} b(r_o) \frac{d\upsilon_o}{dr_o} r_o^{\left(1-\frac{C}{2}\right)} J_p(\sigma_o r_o)\, dr_o \Bigg]
\end{aligned} \tag{4.509}$$

Hence, the solution to the υ_1 problem (*i.e.*, Equation (4.465)), will be

$$\upsilon_1 = f_o + f_1 \cos\theta_o = f_o + \left(\hat{f}_1 + \hat{\hat{f}}_1 \right) \cos\theta_o \tag{4.510}$$

and from this we can get

$$\frac{\partial \upsilon_1}{\partial r_o} = \frac{\partial f_o}{\partial r_o} + \left(\frac{\partial \hat{f}_1}{\partial r_o} + \frac{\partial \hat{\hat{f}}_1}{\partial r_o} \right) \cos\theta_o \tag{4.511}$$

and

$$\frac{\partial \upsilon_1}{\partial \theta_o} = -\left(\hat{f}_1 + \hat{\hat{f}}_1 \right) \sin\theta_o \tag{4.512}$$

which are going to be useful later on.

From our previous work we know that the f_o part of the solution of the υ_1 problem will have no effect on the value of σ_2^2 because it will appear as a multiple of $\cos\theta_o$ and when integrated over $0, 2\pi$ it must vanish. This is important because we do not know the constant $\dot{B}$. Therefore, the f_o part can be dropped from the calculation of σ_2^2.

Now, to find σ_2^2 we must solve Equation (4.386), which is

$$\frac{1}{2}\nabla^2\upsilon_2 + \frac{1}{2}\nabla P_o.\nabla\upsilon_2 + \nabla P_1.\nabla\upsilon_1 + \frac{1}{2}\nabla P_2.\nabla\upsilon_o + \frac{1}{2}\sigma_o^2\upsilon_2 + \sigma_1^2\upsilon_1 + \frac{1}{2}\sigma_2^2\upsilon_o = 0$$

where υ_2 is subject to the two boundary conditions given by Equations (4.387) and (4.388), and they are

$$\upsilon_2(r_o = R_{out}) = 0$$

and

$$\upsilon_2(r_o = 1) = -2R_1\left.\frac{d\upsilon_1}{dr_o}\right|_{r_o=1} - R_1^2\left.\frac{d^2\upsilon_o}{dr_o^2}\right|_{r_o=1} - R_2\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=1}$$

From the earlier work we know that $\sigma_1^2 = 0$. Then, to simplify this problem, let us expand the following terms

$$\nabla P_o \cdot \nabla\upsilon_2 = \vec{i}_r\frac{\partial P_o}{\partial r_o}\cdot\left(\vec{i}_r\frac{\partial\upsilon_2}{\partial r_o} + \frac{\vec{i}_\theta}{r_o}\frac{\partial\upsilon_2}{\partial\theta_o}\right) = \frac{\partial P_o}{\partial r_o}\frac{\partial\upsilon_2}{\partial r_o} \tag{4.513}$$

$$\nabla P_1 \cdot \nabla\upsilon_1 = \left(\vec{i}_r\frac{\partial P_1}{\partial r_o} + \frac{\vec{i}_\theta}{r_o}\frac{\partial P_1}{\partial\theta_o}\right)\cdot\left(\vec{i}_r\frac{\partial\upsilon_1}{\partial r_o} + \frac{\vec{i}_\theta}{r_o}\frac{\partial\upsilon_1}{\partial\theta_o}\right) = \frac{\partial P_1}{\partial r_o}\frac{\partial\upsilon_1}{\partial r_o} + \frac{1}{r_o^2}\frac{\partial P_1}{\partial\theta_o}\frac{\partial\upsilon_1}{\partial\theta_o} \tag{4.514}$$

and

$$\nabla P_2 \cdot \nabla\upsilon_o = \left(\vec{i}_r\frac{\partial P_2}{\partial r_o} + \frac{\vec{i}_\theta}{r_o}\frac{\partial P_2}{\partial\theta_o}\right)\cdot\left(\vec{i}_r\frac{\partial\upsilon_o}{\partial r_o}\right) = \frac{\partial P_2}{\partial r_o}\frac{\partial\upsilon_o}{\partial r_o} \tag{4.515}$$

Then by using Equation (4.392) we can write Equation (4.513) as

$$\nabla P_o \cdot \nabla\upsilon_2 = -\frac{C}{r_o}\frac{\partial\upsilon_2}{\partial r_o} \tag{4.516}$$

and by using Equations (4.392), (4.395) and (4.396), Equation (4.514) becomes

$$\nabla P_1 \cdot \nabla \upsilon_1 = -Cb(r_o)\frac{\partial \upsilon_1}{\partial r_o}\cos\theta_o + Cd(r_o)\frac{\partial \upsilon_1}{\partial \theta_o}\sin\theta_o \tag{4.517}$$

where

$$b(r_o) = \frac{1}{\left(R_{out}^2 - 1\right)}\left(1 + \frac{R_{out}^2}{r_o^2}\right) \tag{4.518}$$

and

$$d(r_o) = \frac{1}{\left(R_{out}^2 - 1\right)}\left(\frac{1}{r_o} - \frac{R_{out}^2}{r_o^3}\right) \tag{4.519}$$

and by using Equations (4.392) and (4.398) we can write Equation (4.515) as

$$\nabla P_2 \cdot \nabla \upsilon_o = -\left[Ce(r_o) + Cf(r_o)\cos 2\theta_o\right]\frac{\partial \upsilon_o}{\partial r_o} \tag{4.520}$$

where

$$e(r_o) = \frac{2}{\left(R_{out}^2 - 1\right)\ln\left(R_{out}\right)r_o} \tag{4.521}$$

and

$$f(r_o) = \frac{2}{\left(R_{out}^2 - 1\right)^2}\left(\frac{R_{out}^4}{r_o^3} + r_o\right) \tag{4.522}$$

Hence, by substituting Equations (4.516), (4.517) and (4.520) into Equation (4.386) we obtain

$$\begin{aligned} &\frac{\partial^2 \upsilon_2}{\partial r_o^2} + \frac{(1-C)}{r_o}\frac{\partial \upsilon_2}{\partial r_o} + \frac{1}{r_o^2}\frac{\partial^2 \upsilon_2}{\partial \theta_o^2} - 2Cb(r_o)\frac{\partial \upsilon_1}{\partial r_o}\cos\theta_o \\ &\qquad + 2Cd(r_o)\frac{\partial \upsilon_1}{\partial \theta_o}\sin\theta_o - Ce(r_o)\frac{\partial \upsilon_o}{\partial r_o} \\ &\qquad\qquad - Cf(r_o)\frac{\partial \upsilon_o}{\partial r_o}\cos 2\theta_o + \sigma_o^2 \upsilon_2 + \sigma_2^2 \upsilon_o = 0 \end{aligned} \tag{4.523}$$

where the general solution of υ_2 can take the following form

$$\upsilon_2 = g_o + g_1 \cos\theta_o + g_2 \cos 2\theta_o \tag{4.524}$$

Then, by substituting Equations (4.524) and (4.448) into Equation (4.523) and into its boundary conditions (*i.e.*, Equations (4.387) and (4.388)), we get the following set of problems[18]: First for g_o, the problem is

$$\frac{d^2 g_o}{dr_o^2} + \frac{(1-C)}{r_o}\frac{dg_o}{dr_o} + \sigma_o^2 g_o - Cb(r_o)\left(\frac{d\hat{f}_1}{dr_o} + \frac{d\hat{\hat{f}}_1}{dr_o}\right) - Cd(r_o)\left(\hat{f}_1 + \hat{\hat{f}}_1\right)$$
$$- Ce(r_o)\frac{d\upsilon_o}{dr_o} + \sigma_2^2 \upsilon_o = 0 \tag{4.525}$$

where

$$g_o(r_o = R_{out}) = 0 \tag{4.526}$$

and

$$g_o(r_o = 1) = -\left(\left.\frac{d\hat{f}_1}{dr_o}\right|_{r_o=1} + \left.\frac{d\hat{\hat{f}}_1}{dr_o}\right|_{r_o=1}\right) + \left(1 - \frac{C}{2}\right)\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=1} \tag{4.527}$$

For g_1, the problem is

$$\frac{d^2 g_1}{dr_o^2} + \frac{(1-C)}{r_o}\frac{dg_1}{dr_o} + \left(\sigma_o^2 - \frac{1}{r_o^2}\right) g_1 = 0 \tag{4.528}$$

where

$$g_1(r_o = R_{out}) = 0 \tag{4.529}$$

and

$$g_1(r_o = 1) = 0 \tag{4.530}$$

For g_2, the problem is

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[18] Note that the f_o solution of the υ_1 problem is dropped from this calculation as it vanishes due to the reasons explained earlier.

$$\frac{d^2 g_2}{dr_o^2}+\frac{(1-C)}{r_o}\frac{dg_2}{dr_o}+\left(\sigma_o^2-\frac{4}{r_o^2}\right)g_2-Cb(r_o)\left(\frac{d\hat{f}_1}{dr_o}+\frac{d\hat{\hat{f}}_1}{dr_o}\right) \\ +2Cd(r_o)\left(\hat{f}_1+\hat{\hat{f}}_1\right)\sin\theta_o-Cf(r_o)\frac{d\upsilon_o}{dr_o}=0 \qquad (4.531)$$

where

$$g_2(r_o=R_{out})=0 \qquad (4.532)$$

and

$$g_2(r_o=1)=-\left(\left.\frac{d\hat{f}_1}{dr_o}\right|_{r_o=1}+\left.\frac{d\hat{\hat{f}}_1}{dr_o}\right|_{r_o=1}\right)-\frac{C}{2}\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=1} \qquad (4.533)$$

Notice that σ_2^2 appears only in the g_o problem (*i.e.*, Equation (4.525)), therefore to obtain σ_2^2 we must use Equation (4.525). To begin, let us rewrite Equation (4.525) in self-adjoint form by multiplying it by the function $r_o^{(1-C)}$, which gives

$$r_o^{(1-C)}\frac{d^2 g_o}{dr_o^2}+(1-C)r_o^{(-C)}\frac{dg_o}{dr_o}+\sigma_o^2 r_o^{(1-C)}g_o-Cr_o^{(1-C)}b(r_o)\left(\frac{d\hat{f}_1}{dr_o}+\frac{d\hat{\hat{f}}_1}{dr_o}\right) \\ -Cr_o^{(1-C)}d(r_o)\left(\hat{f}_1+\hat{\hat{f}}_1\right)-Cr_o^{(1-C)}e(r_o)\frac{d\upsilon_o}{dr_o}+\sigma_2^2 r_o^{(1-C)}\upsilon_o=0 \qquad (4.534)$$

which can be written as

$$\frac{d}{dr_o}\left(r_o^{(1-C)}\frac{dg_o}{dr_o}\right)+\sigma_o^2 r_o^{(1-C)}g_o-Cr_o^{(1-C)}b(r_o)\left(\frac{d\hat{f}_1}{dr_o}+\frac{d\hat{\hat{f}}_1}{dr_o}\right) \\ -Cr_o^{(1-C)}d(r_o)\left(\hat{f}_1+\hat{\hat{f}}_1\right)-Cr_o^{(1-C)}e(r_o)\frac{d\upsilon_o}{dr_o}+\sigma_2^2 r_o^{(1-C)}\upsilon_o=0 \qquad (4.535)$$

Then, multiply Equation (4.535) by υ_o, which gives

$$\begin{aligned}\upsilon_o \frac{d}{dr_o}\left(r_o^{(1-C)}\frac{dg_o}{dr_o}\right)+\sigma_o^2 r_o^{(1-C)}\upsilon_o g_o - C r_o^{(1-C)} b(r_o)\upsilon_o\left(\frac{d\hat{f}_1}{dr_o}+\frac{d\hat{\hat{f}}_1}{dr_o}\right)\\ - C r_o^{(1-C)} d(r_o)\upsilon_o\left(\hat{f}_1+\hat{\hat{f}}_1\right) - C r_o^{(1-C)} e(r_o)\upsilon_o\frac{d\upsilon_o}{dr_o}+\sigma_2^2 r_o^{(1-C)}\upsilon_o^2=0\end{aligned}\tag{4.536}$$

and multiply Equation (4.458) by g_o, to get

$$g_o \frac{d}{dr_o}\left(r_o^{(1-C)}\frac{d\upsilon_o}{dr_o}\right)+\sigma_o^2 r_o^{(1-C)} g_o \upsilon_o=0\tag{4.537}$$

Then, subtract Equation (4.537) from Equation (4.536), to obtain

$$\begin{aligned}\upsilon_o \frac{d}{dr_o}\left(r_o^{(1-C)}\frac{dg_o}{dr_o}\right)-g_o \frac{d}{dr_o}\left(r_o^{(1-C)}\frac{d\upsilon_o}{dr_o}\right)=C r_o^{(1-C)} b(r_o)\upsilon_o\left(\frac{d\hat{f}_1}{dr_o}+\frac{d\hat{\hat{f}}_1}{dr_o}\right)\\ + C r_o^{(1-C)} d(r_o)\upsilon_o\left(\hat{f}_1+\hat{\hat{f}}_1\right) + C r_o^{(1-C)} e(r_o)\upsilon_o\frac{d\upsilon_o}{dr_o}-\sigma_2^2 r_o^{(1-C)}\upsilon_o^2\end{aligned}\tag{4.538}$$

Now integrate Equation (4.538) over the reference domain, which gives

$$\begin{aligned}\int_1^{R_{out}}\upsilon_o \frac{d}{dr_o}\left(r_o^{(1-C)}\frac{dg_o}{dr_o}\right)dr_o-\int_1^{R_{out}}g_o \frac{d}{dr_o}\left(r_o^{(1-C)}\frac{d\upsilon_o}{dr_o}\right)dr_o=\\ \int_1^{R_{out}} C r_o^{(1-C)} b(r_o)\upsilon_o\left(\frac{d\hat{f}_1}{dr_o}+\frac{d\hat{\hat{f}}_1}{dr_o}\right)dr_o+\int_1^{R_{out}} C r_o^{(1-C)} d(r_o)\upsilon_o\left(\hat{f}_1+\hat{\hat{f}}_1\right)dr_o\\ +\int_1^{R_{out}} C r_o^{(1-C)} e(r_o)\upsilon_o\frac{d\upsilon_o}{dr_o}dr_o-\int_1^{R_{out}}\sigma_2^2 r_o^{(1-C)}\upsilon_o^2 dr_o\end{aligned}\tag{4.539}$$

Then, by using the integration by part on the left-hand side of Equation (4.539) we obtain

$$\begin{aligned}\upsilon_o r_o^{(1-C)}\frac{dg_o}{dr_o}\Bigg|_1^{R_{out}}-g_o r_o^{(1-C)}\frac{d\upsilon_o}{dr_o}\Bigg|_1^{R_{out}}=\int_1^{R_{out}} C r_o^{(1-C)} b(r_o)\upsilon_o\left(\frac{d\hat{f}_1}{dr_o}+\frac{d\hat{\hat{f}}_1}{dr_o}\right)dr_o\\ +\int_1^{R_{out}} C r_o^{(1-C)} d(r_o)\upsilon_o\left(\hat{f}_1+\hat{\hat{f}}_1\right)dr_o+\int_1^{R_{out}} C r_o^{(1-C)} e(r_o)\upsilon_o\frac{d\upsilon_o}{dr_o}dr_o-\int_1^{R_{out}}\sigma_2^2 r_o^{(1-C)}\upsilon_o^2 dr_o\end{aligned}\tag{4.540}$$

Then, because υ_o and g_o vanish at $r_o = R_{out}$ and because υ_o vanishes at $r_o = 1$ and by using Equations (4.527), Equation (4.540) reduces to

$$\begin{aligned}-\left(\left.\frac{d\hat{f}_1}{dr_o}\right|_{r_o=1}+\left.\frac{d\hat{\hat{f}}_1}{dr_o}\right|_{r_o=1}\right)\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=1}+\left(1-\frac{C}{2}\right)\left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=1} &= \\ &\int_1^{R_{out}} C r_o^{(1-C)} b(r_o)\upsilon_o\left(\frac{d\hat{f}_1}{dr_o}+\frac{d\hat{\hat{f}}_1}{dr_o}\right)dr_o \\ &+\int_1^{R_{out}} C r_o^{(1-C)} d(r_o)\upsilon_o\left(\hat{f}_1+\hat{\hat{f}}_1\right)dr_o \\ &+\int_1^{R_{out}} C r_o^{(1-C)} e(r_o)\upsilon_o\frac{d\upsilon_o}{dr_o}dr_o-\int_1^{R_{out}}\sigma_2^2 r_o^{(1-C)}\upsilon_o^2 dr_o\end{aligned} \tag{4.541}$$

which can be arranged and rewritten as

$$\begin{aligned}\sigma_2^2=\Bigg[&\left.\frac{d\hat{f}_1}{dr_o}\right|_{r_o=1}\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=1}+\left.\frac{d\hat{\hat{f}}_1}{dr_o}\right|_{r_o=1}\left.\frac{d\upsilon_o}{dr_o}\right|_{r_o=1}-\left(1-\frac{C}{2}\right)\left(\frac{d\upsilon_o}{dr_o}\right)^2_{r_o=1} \\ &+C\int_1^{R_{out}} r_o^{(1-C)} b(r_o)\upsilon_o\left(\frac{d\hat{f}_1}{dr_o}+\frac{d\hat{\hat{f}}_1}{dr_o}\right)dr_o+C\int_1^{R_{out}} r_o^{(1-C)} d(r_o)\upsilon_o\left(\hat{f}_1+\hat{\hat{f}}_1\right)dr_o \\ &+C\int_1^{R_{out}} r_o^{(1-C)} e(r_o)\upsilon_o\frac{d\upsilon_o}{dr_o}dr_o\Bigg]\Big/\int_1^{R_{out}} r_o^{(1-C)}\upsilon_o^2 dr_o\end{aligned} \tag{4.542}$$

This is our equation for σ_2^2 and it has been obtained without solving for υ_2. Notice that all the components of σ_2^2 depend on C and on r_o.

Now to decide about the sign of σ_2^2 we must find the sign of the right-hand side of Equation (4.542). But in this case, the sign of the right-hand side of Equation (4.542) can only be found numerically. Then, to achieve this, let us begin by holding R_{out} fixed and change the value of C to see how the strength of the radial flow and, consequently, the convection affects the correction factor σ_2^2/σ_o^2. This calculation is presented as graphs in Figure 4-13 for $C>0$, outward flow, and in Figure 4-14 for $C<0$, inward flow. Then,

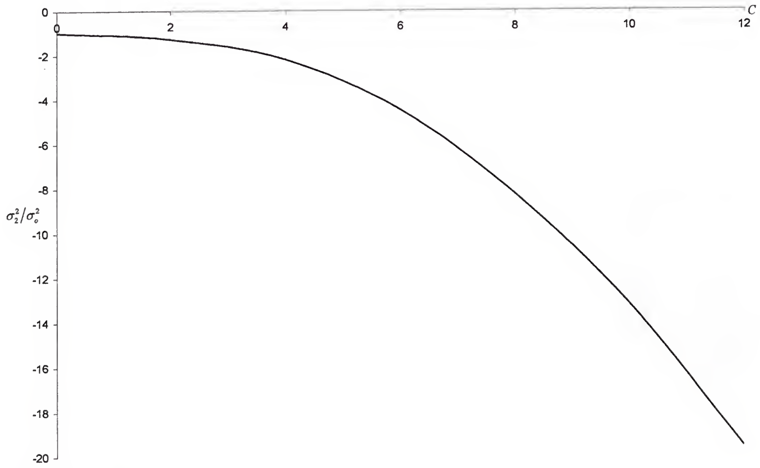

Figure 4-13. Values of σ_2^2/σ_o^2 at $R_{out} = 5$ and different values of C's > 0. (Constant rod and pipe temperatures)

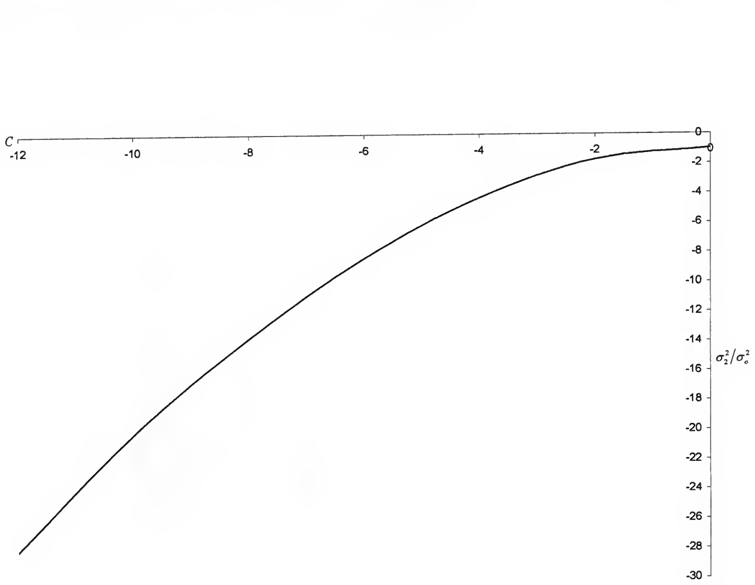

Figure 4-14. Values of σ_2^2/σ_o^2 at $R_{out} = 5$ and different values of C's < 0. (Constant rod and pipe temperatures)

let us hold C fixed and play with the values of R_{out} to see how the thickness of the region affects the results. This calculation is presented as graph in Figure 4-15.

From these three figures we observe that the values of σ_2^2/σ_o^2 are always less than zero. Hence, the sign of σ_2^2 is always negative regardless of the strength of the radial flow, its direction and the thickness of the annular region. This is happened because, as we explained earlier, as we shift the rod the radial velocity across the thick region to the left of the rod become weaker and as a result of this the redial flow cannot push the maximum temperature, which lies near the middle of the region, closer to one of the two heat sinks. Therefore, the temperature of the thick region will increase and, then, more heat will be produced. Because of this the effect of having thick heat-producing region will conquer the stabilizing effects of the other factors (*i.e.*, the radial flow and the heat conduction and the fluid flow in the azimuthal direction), and the system become thermally less stable.

From Figure 4-13 and Figure 4-14 we can observe that as the strength of the radial flow increases (*i.e.*, high flow), the value of σ_2^2/σ_o^2 decreases. But we now know that in the case of high flow the correction factor σ_2^2/σ_o^2 cannot be important. This is because as the strength of the radial flow increases, regardless of its direction, it will push the maximum temperature more and more closer to one of the two heat sinks and above certain value of C the maximum temperature will be on one of the two heat sinks and in such a case the temperature of the annular region will be equal to the temperature of the incoming fluid. Therefore, this problem must always remain stable regardless of the value of σ_2^2/σ_o^2. To prove this turn to the eigenvalue problem (*i.e.*, Equation (4.368)), namely

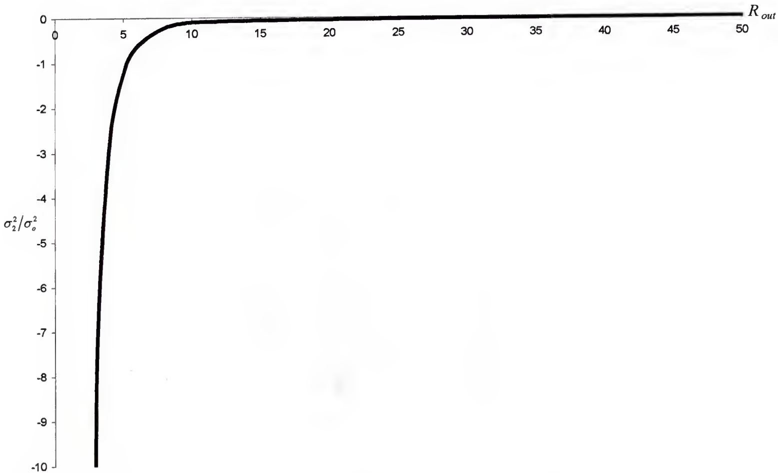

Figure 4-15. Values of σ_2^2/σ_o^2 at C=2 and different values of R_{out}. (Constant rod and pipe temperatures)

$$\nabla^2 \upsilon + \nabla P . \nabla \upsilon + \sigma^2 \upsilon = 0$$

and use Equation (4.365) to rewrite this equation as

$$\nabla^2 \upsilon - \vec{V} . \nabla \upsilon + \sigma^2 \upsilon = 0 \tag{4.543}$$

where υ must satisfy

$$\upsilon\left(r = R_{out}\right) = 0$$

and

$$\upsilon\left(r = R(\theta, \varepsilon)\right) = 0$$

Then, let us multiply Equation (4.543) by υ, to obtain

$$\upsilon \nabla^2 \upsilon - \vec{V} . \upsilon \nabla \upsilon + \sigma^2 \upsilon^2 = 0 \tag{4.544}$$

but we know that

$$\upsilon \nabla \upsilon = \nabla\left(\frac{1}{2} \upsilon^2\right) \tag{4.545}$$

then Equation (4.544) will become

$$\upsilon \nabla^2 \upsilon - \vec{V} . \nabla\left(\frac{1}{2} \upsilon^2\right) + \sigma^2 \upsilon^2 = 0 \tag{4.546}$$

Now, by using the following formula[19]

$$\vec{V} . \nabla\left(\frac{1}{2} \upsilon^2\right) = \nabla . \left(\frac{1}{2} \upsilon^2 \vec{V}\right) - \frac{1}{2} \upsilon^2 \left(\nabla . \vec{V}\right) \tag{4.547}$$

and by using the equation of continuity ($\nabla . \vec{V} = 0$), we can rewrite Equation (4.546) as

$$\upsilon \nabla^2 \upsilon - \nabla . \left(\frac{1}{2} \upsilon^2 \vec{V}\right) + \sigma^2 \upsilon^2 = 0 \tag{4.548}$$

where

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[19] Granger, R.A., 1995, Fluid Mechanics, Dover Publications, Inc., New York.

$$\upsilon\nabla^2\upsilon = \nabla\cdot(\upsilon\nabla\upsilon) - \nabla\upsilon\cdot\nabla\upsilon \tag{4.549}$$

Then, substitute Equation (4.549) into Equation (4.548), to obtain

$$\nabla\cdot(\upsilon\nabla\upsilon) - (\nabla\upsilon)^2 - \nabla.\left(\frac{1}{2}\upsilon^2\vec{V}\right) + \sigma^2\upsilon^2 = 0 \tag{4.550}$$

Now, let us integrate Equation (4.550) over the reference domain and then use the divergence theorem, to get

$$\begin{aligned} -\sigma^2\int_R^{R_{out}}\int_0^{2\pi}\upsilon^2 d\theta r dr + \int_R^{R_{out}}\int_0^{2\pi}(\nabla\upsilon)^2 d\theta r dr &= \int_0^{2\pi} R_{out}\left(\upsilon\frac{d\upsilon}{dr} - \frac{1}{2}\upsilon^2 V_r\right)_{r=R_{out}} d\theta \\ &+ \int_0^{2\pi} R\left(-\upsilon\frac{d\upsilon}{dr} + \frac{1}{2}\upsilon^2 V_r\right)_{r=R} d\theta \end{aligned} \tag{4.551}$$

The right-hand side of this equation is zero because υ vanishes at $r = R$ and at $r = R_{out}$. Then, Equation (4.551) reduces to

$$\sigma^2 = \frac{\int_R^{R_{out}}\int_0^{2\pi}(\nabla\upsilon)^2 d\theta r dr}{\int_R^{R_{out}}\int_0^{2\pi}\upsilon^2 d\theta r dr} > 0 \tag{4.552}$$

which is always greater than zero. Thus, we can observe form Equation (4.552) that σ^2 is always positive and, therefore, this problem, the distorted annulus with radial flow, is always stable. Based on this we can conclude that, in the perturbed annulus case with radial flow the correction to σ^2 cannot be important. From this we learn that the term $\varepsilon^2\sigma_2^2/2$, which appears in Equation (4.378), must be small not just ε^2.

Also in Figure 4-13 and Figure 4-14 we notice that the values of σ_2^2/σ_o^2 in the case of outward flow, $C > 0$, are greater than those for the inward flow, $C < 0$. This means that the system with outward flow is more stable than that with the inward flow. This might be due to the fact that for this case the maximum temperature lies always somewhere in the middle of the region but closer to the outer wall, then it become much easier for the outward flow to push the maximum temperature closer to the outer wall, outer heat sink, other than for the inward flow to bring the maximum temperature closer to the inner wall, inner heat sink.

In Figure 4-15 we can see that as the value of R_{out} increases the value of σ_2^2/σ_o^2 decreases. This is an expected result because of two reasons. The first reason is that, as R_{out} increases the thickness of heat-producing region increases and from our earlier work we know that thick regions are bad heat conductor. The second reason is due to the effect of the displacement on the radial velocity. Because, the radial velocity across thick regions becomes weaker (*i.e.*, the displacement slows the response of the radial flow), and, therefore, the radial flow will not be able to push the maximum temperature close enough to one of the two sinks. Thus, the maximum temperature will remain somewhere near the middle of the region and, then, the temperature of the thick region will increase, but the higher the temperature the higher the heat to be generated. Because of these two reasons the effect of having thick heat-producing region will defeat the stabilizing effects of the other factors (*i.e.*, the radial flow and the heat conduction and the fluid flow in the azimuthal direction), and the system become thermally less stable.

Thus far all the components of σ^2 (*i.e.*, σ_o^2, σ_1^2 and σ_2^2), are found. The next step is to determine the value of σ^2 for the distorted annulus, which is defined as

$$\sigma^2 = \sigma_o^2 + \varepsilon\sigma_1^2 + \frac{\varepsilon^2}{2}\sigma_2^2 + \ldots$$

and then compare its value with that at the reference state (*i.e.*, σ_o^2). To do this, recall that, σ_o^2 and ε are positive quantities and σ_1^2 is zero while σ_2^2 is a negative quantity. Then we can see that σ^2 for the distorted annulus is smaller than it is for the reference state (*i.e.*, the perturbation decreases the value of σ^2). This is because the radial flow, the heat conduction and the fluid flow in the azimuthal direction could not overcome the effect of the wider region to the left of the rod due to the reasons explained earlier.

Based on this we can conclude that, at constant walls temperatures, a symmetric annular region is thermally more stable than a distorted one. Therefore, the perturbation in this case decreases the thermal stability of the annular region.

It is important to mention that, if we solve this problem again but without perturbing the flow then the values of σ_o^2 will be the same as for the previous case but σ_2^2 will take the following form[20]

$$\sigma_2^2 = \left[\left. \frac{d\hat{f}_1}{dr_o} \right|_{r_o=1} \left. \frac{d\upsilon_o}{dr_o} \right|_{r_o=1} - \left(1 - \frac{C}{2}\right) \left(\frac{d\upsilon_o}{dr_o} \right)^2_{r_o=1} \right] / \int_1^{R_{out}} r_o^{(1-C)} \upsilon_o^2 dr_o \qquad (4.553)$$

Then, to decide the sign of σ_2^2 we must find the sign of the right-hand side of this equation and to achieve this, let us hold R_{out} fixed and change the value of C to see how the strength of the radial flow and, thus, the convection affects the correction factor σ_2^2/σ_o^2. This calculation is presented as graphs in Figure 4-16 for $C > 0$, outward flow, and in Figure 4-17 for $C < 0$, inward flow. Now observe that Figure 4-16 and Figure 4-17 are identical regardless of the direction of the flow (*i.e.*, for any given value of

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[20] This formula can be obtained from Equation (4.542) where all the terms that depend on the perturbed flow are canceled.

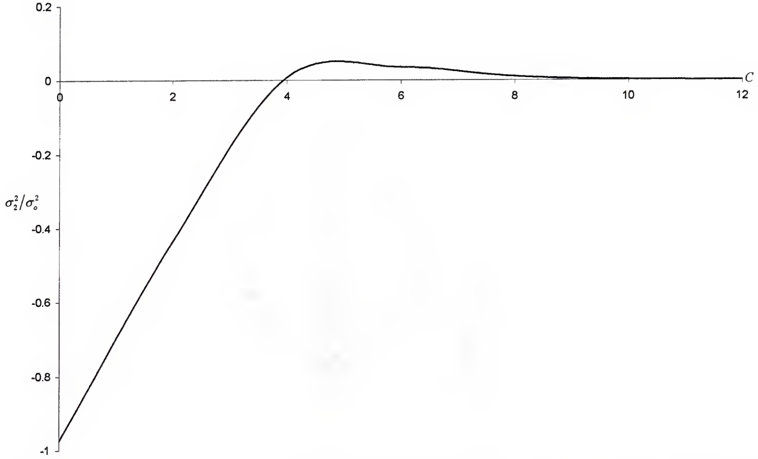

Figure 4-16. Values of σ_2^2/σ_o^2 at $R_{out} = 5$ and different values of C's > 0. (Isothermal inner wall, unperturbed flow)

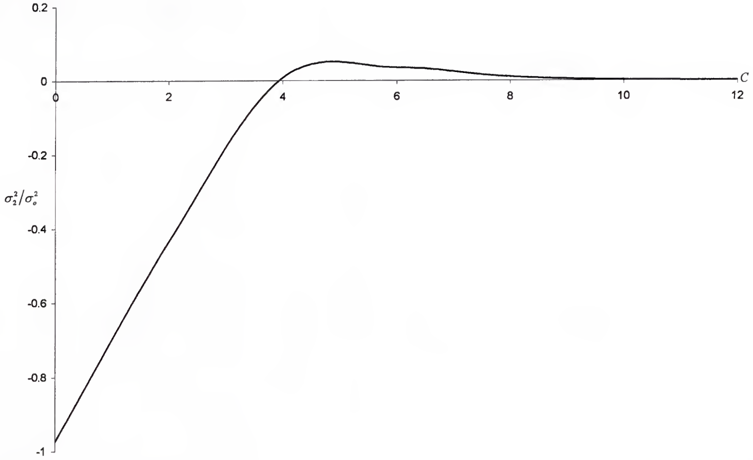

Figure 4-17. Values of σ_2^2/σ_o^2 at $R_{out} = 5$ and different values of C's < 0. (Isothermal inner wall, unperturbed flow)

C the value of σ_2^2/σ_o^2 is the same regardless of the sign of C). Therefore, the thermal stability of this problem is independent of the direction of the flow. Thus, the effect of the magnitude of C on the thermal stability of this case can be seen from any of the two figures (Figure 4-16 or Figure 4-17).

Then on considering Figure 4-16 we can see that the sign of σ_2^2/σ_o^2 is negative for low values of C, at low flow, but it becomes positive as the strength of the flow increases. Also we can observe that, the values of σ_2^2/σ_o^2 will increase as C increases (*i.e.*, as the strength of the flow increases), but at high flow, $C > 5$, the values of σ_2^2/σ_o^2 will begin to decrease until it becomes close to zero. This is happened because as the strength of the flow increases the maximum temperature will be pushed closer to one of the two heat sinks depending on the direction of the flow and this will increases the thermal stability but at high radial flow the maximum temperature will be located on one of the two heat sinks and then the temperature in the annular region will become equal to the temperature of the incoming fluid and, thus, the correction of σ^2 cannot be important. Based on this we can conclude that, at constant temperature walls with unperturbed flow, a distorted annular region is thermally more stable than a symmetric one. Therefore, the perturbation in this case increases the thermal stability of the annular region.

Now, let us summarized what we have done and learned in this chapter. The aim of this chapter was to study the effect of perturbing the inner cylinder on the thermal stability of a heat-generating annular region and to show how this stability depends on the boundary conditions and the heat-transfer modes. To achieve this, we started by solving a simpler problem, the thermal stability of a perturbed circular disk, through which we got familiar with the cylindrical coordinates and we learned that the thermal stability of a circular disk increases as it is perturbed, deformed, into an elliptical disk. This was a useful result as it confirms our earlier conclusion in Chapter 2 that is: thermal stability can be affected by a geometry change. Then, we came back to the proposed problem (*i.e.*, the thermal stability of a heat-generating annular region), and we began by considering the case where the heat produced in the domain is transferred only by conduction (*i.e.*, no flow). For this case the proposed problem was solved for three different boundary conditions for the inner wall while the boundary condition at the outer wall was assumed to be isothermal at all cases. Doing this we found that, at constant wall temperatures and at adiabatic inner wall, the annular region becomes thermally less stable as the inner cylinder is shifted from the center of the geometry, but at the perfectly conducting inner wall we found the opposite (*i.e.*, the distorted annular region is thermally more stable than a symmetric one). Then, a comparison between the results obtained from these three cases was made to show the difference and from this we found that, at the reference state, the region with constant wall temperatures is thermally more stable than the regions with adiabatic and perfectly conducting inner wall. Also we found that the region with adiabatic inner wall and the region with perfectly conducting inner wall are equally stable. In addition to that, again at the reference case, we found that thick heat-producing annulus is thermally less stable than thin heat-producing annulus. This result agrees with our earlier conclusion in Chapter 3 that is: thick heat-producing regions are thermally less stable than thin heat-producing regions. Then to show the effect of perturbing the inner cylinder on the thermal stability of these problems a comparison between the results obtained from the distorted annulus cases was made. From this comparison we learned that, regardless of the reduction in its thermal stability due to the displacement, the region with constant walls temperatures is always more stable than the other two cases. Also we found that, in thin regions, the region with adiabatic inner wall is slightly less stable than the region with perfectly conducting inner wall. Furthermore, we found that the thermal stability of the region with perfectly conducting inner wall is weakly dependent on the location of the inner cylinder. Again, these results completely agree with our previous work in Chapter 3 that is: when the central layer is displaced, the critical value of δ for the case of perfectly conducting central layer does not change while it decreases for the case of adiabatic central layer. From this we learned that, the thermal stability of a heat-generating annular region depends strongly on the location of the inner cylinder and on the boundary conditions of the region. Thus, in designing heat exchangers, which typically designed in concentric cylinder form, the designer must be very careful to get the alignment right and to take in his consideration the boundary conditions of the system. Then, we considered the case where the heat produced in the domain is carried to the walls by both conduction and convection. To study this case we added a radial flow to one of the three problems solved earlier, in this work the problem of constant walls temperatures was chosen, and then we assumed that the fluid flow is described via Darcy's law. As usual, we started by solving a simpler problem, the case of parallel plane walls, to get some guidance. Then we came back to the proposed problem and we started to solve it. Doing this we found that, at the reference state, the radial flow, regardless of its direction, increase the thermal stability of this problem (*i.e.*, thermal stability enhanced when heat transfer by convection comes into action). But on perturbing the inner cylinder the thermal stability decreases and it becomes function in the direction of the radial flow, that is: the case with outward flow is thermally more stable than the case with inward flow. Also, we found that at high flow the correction factors cannot be important and the problem remains always stable. In addition to this we were able to show by using simple derivation that this problem must always be stable regardless of the direction of the flow or its strength. From this we convinced our selves that the thermal stability of this problem strongly depends on the heat-transfer mode.

In the next chapter (Chapter 5), another practical problem is considered. This problem is to study the effect of moving boundaries on the thermal stability of a distorted annulus.

CHAPTER 5
VISCOUS HEATING OF LIQUIDS IN SIMPLE FLOWS

In this chapter we will illustrate how the results presented earlier tie into some concrete physical problems. Our main example is based on the fact that the viscosity of a liquid is a decreasing function of the temperature of the liquid. In some flows the stress is constant throughout the flowing fluid and because of this the rate of shearing is inversely proportional to the viscosity of the fluid. At the same time the rate of frictional heating is given by the product of the viscosity and the square of the rate of shearing and hence is inversely proportional to the viscosity of the liquid. Due to this, the rate of frictional heating is an increasing function of the temperature of the liquid and is thus a problem of interest to us. We will begin by explaining how viscous flow problems are in fact thermal ignition problems[1] and are thermal ignition problems in which the heat source depends linearly on temperature. We can illustrate this connection in certain simple problems. The analogy is precise in the base problems and only begins to fall apart when the geometry perturbation induces a flow perturbation as well as a temperature perturbation. These problems are presented and discussed in the following section.

5.1 The Temperature Equation in Viscous Fluid Flow Problems

In this section we will discuss three simple viscous flow problems to show that their temperature equations are similar to those in thermal ignition problems for which the heat-generation function depends linearly on temperature. These three problems are:

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[1] The heat source in thermal ignition problems is the exothermic heat of reaction while in viscous fluid flow problems it is the viscous heating.

Couette flow between parallel plane walls, Poiseuille flow between parallel plane walls and Couette flow between coaxial cylinders (*i.e.*, between a rod and a pipe). In these problems, we shall consider the fluid flow to be *laminar* and *incompressible*. These three problems are presented and discussed in the following sub-sections.

5.1.1 Couette Flow between Parallel Plane Walls

Let us consider a Newtonian fluid confined in the space between two infinite parallel plane walls separated by a distance $2L$, as depicted in Figure 5-1. The upper wall moves at a velocity V in the z-direction while the lower wall is at rest. In this case, the fluid will flow only in the z-direction due to the movement of the upper wall ($V_x = 0$). Because of this flow, frictional heat is produced as a result of shear between adjacent layers of the fluid. The heat generated in the fluid is lost to the boundaries, which are held at a fixed temperature denoted T_w. The reciprocal of the viscosity is called the fluidity and for many fluids the fluidity depends linearly on the temperature of the fluid over a fair temperature range.[2] Thus, the fluidity can be defined as $\mu_w/\mu = 1 + s(T - T_w)$ where μ_w denotes $\mu(T_w)$ and where s describes the sensitivity of the fluidity to the temperature. The thermal conductivity of the fluid, k, is assumed to be constant and the flow takes place entirely in the z-direction at no pressure gradient (*i.e.*, $dp/dz = 0$). The gravity can be neglected.

To keep this system stable the heat generated needs to be removed from the system at the rate it is produced. This heat is removed by conduction to the two walls which serve as heat sinks. To conduct the heat away the temperature of the fluid must rise above that of the two walls. And the more distant a point of the fluid is from the walls the higher its temperature must rise. Therefore, the maximum temperature is expected to be in the middle of the flow domain (*i.e.*, at $x = 0$).

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[2] Subrahmaniam, N., 2000, Frictional Heating in Shear Flows, Ph.D. Dissertation, University of Florida, Gainesville.

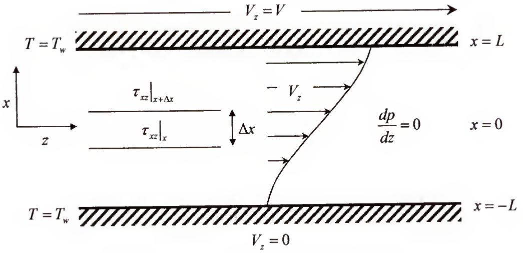

Figure 5-1. Couette flow between parallel plane walls, $dp/dz = 0$.

The steady temperature rise in this problem, referring to Figure 5-1 for a definition of the variables, can be described by the thermal energy equation

$$k\frac{d^2T}{dx^2} + \mu\left(\frac{dV_z}{dx}\right)^2 = 0 \tag{5.1}$$

where T is the temperature field and where V_z is the z-component of the fluid velocity vector. The boundary conditions to be satisfied are

$$T(x = L) = T_w \tag{5.2}$$

$$T(x = -L) = T_w \tag{5.3}$$

$$V_z(x = L) = V \tag{5.4}$$

and

$$V_z(x=-L)=0 \tag{5.5}$$

The shear stress τ_{xz} must satisfy

$$\frac{d\tau_{xz}}{dx}=0 \tag{5.6}$$

which tells us that the shear stress is constant across the gap, which means

$$\tau_{xz}=\mu\frac{dV_z}{dx}=C \tag{5.7}$$

where C is a constant. From this equation we find

$$\frac{dV_z}{dx}=\frac{\tau_{xz}}{\mu}=\frac{C}{\mu} \tag{5.8}$$

and hence Equation (5.1) becomes

$$k\frac{d^2T}{dx^2}+\frac{C^2}{\mu}=0 \tag{5.9}$$

Now, typically, the viscosity of a liquid decreases with increasing temperature. Therefore, as the fluid temperature rises due to the frictional heating the fluid viscosity will decrease and, therefore, the fluidity, which is the reciprocal of the viscosity, will increase. And according to Equation (5.8), as the fluidity increases the velocity gradient increases and, consequently, the stronger the heat source becomes.

In this form we see that, the temperature equation of the Couette flow problem (*i.e.*, Equation (5.9)), is just the temperature equation of the thermal ignition problem of parallel plane walls which was discussed and solved earlier in Chapter 3, namely

$$k\frac{d^2T}{dx^2}+Q(T)=0 \tag{5.10}$$

where $Q(T)$ denotes the rate at which heat is produced in the fluid per unit volume, assumed to be linear function in T (*i.e.*, $Q(T)=Q_o\left[1+S(T-T_w)\right]$), where S is the sensitivity of heat generation to temperature.

Next we see what happens if the flow is driven by a pressure gradient, with the upper wall held fixed.

5.1.2 Poiseuille Flow between Parallel Plane Walls

The geometry is as before and is depicted in Figure 5.2. In this case, the upper and the lower walls are at rest and the fluid will flow in the z-direction due to a pressure gradient, which is assumed to be a constant less than zero.

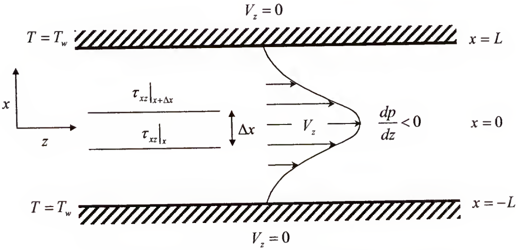

Figure 5-2. Poiseuille flow between parallel plane walls, $dp/dz<0$.

The steady temperature rise in this problem, referring to Figure 5-2 for a definition of the variables, can be described by Equation (5.1), namely

$$k\frac{d^2T}{dx^2}+\mu\left(\frac{dV_z}{dx}\right)^2=0$$

But in this case the boundary conditions to be satisfied are

$$T\left(x=L\right)=T_w \tag{5.11}$$

$$T\left(x=-L\right)=T_w \tag{5.12}$$

$$V_z\left(x=L\right)=0 \tag{5.13}$$

and

$$V_z\left(x=-L\right)=0 \tag{5.14}$$

Under the conditions of this problem, the shearing stress must satisfy

$$\frac{\partial \tau_{xz}}{\partial x}=\frac{\partial p}{\partial z}=-\mathbb{P} \tag{5.15}$$

where $\mathbb{P}$ is a positive constant, whence τ_{xz} is

$$\tau_{xz}=\mu\frac{\partial V_z}{\partial x}=-\mathbb{P}x \tag{5.16}$$

whereupon

$$\frac{dV_z}{dx}=\frac{\tau_{xz}}{\mu}=-\frac{\mathbb{P}x}{\mu} \tag{5.17}$$

and Equation (5.1) becomes

$$k\frac{d^2T}{dx^2}+\frac{\left(\mathbb{P}x\right)^2}{\mu}=0 \tag{5.18}$$

Again we have our positive feed back (*i.e.*, the higher T is the stronger the heat source becomes), but now there is an added complication: the heat source depends on x as well as upon T, being higher at the wall due to the x term where it is lower due to T and lower in the center due to the x term where it is higher due to T.

This problem might then be thought to be more stable than the previous problem, all other things being equal. Now let us turn to a cylindrical geometry.

5.1.3 Couette Flow between Coaxial Cylinders

Let the fluid lie in the space between a rod of radius R_{in} and a pipe of radius R_{out}, as depicted in Figure 5-3. The pipe rotates at a velocity ΩR_{out} in the θ-direction while the rod is at rest. In this case, the fluid will flow only in the θ-direction because of the rotation of the pipe (*i.e.*, $V_r = 0$). Again heat is produced as a result of shear between adjacent layers of the fluid and this heat must be conducted to the boundaries, which are held at fixed temperatures denoted T_{rod} and T_{pipe}, see Figure 5-3.

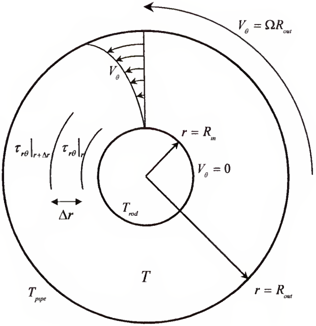

Figure 5-3. Couette flow between coaxial cylinders.

The steady temperature rise in this problem, referring to Figure 5-3 for a definition of the variables, can be described by the thermal energy equation

$$k\frac{1}{r}\frac{d}{dr}(r\frac{dT}{dr})+\mu\left[r\frac{d}{dr}\left(\frac{V_\theta}{r}\right)\right]^2=0 \tag{5.19}$$

where T is assumed to depend only on r and where V_θ is the θ -component of the fluid velocity vector. The boundary conditions to be satisfied are

$$T\left(r = R_{in}\right) = T_{rod} \tag{5.20}$$

$$T\left(r = R_{out}\right) = T_{pipe} \tag{5.21}$$

$$V_\theta\left(r = R_{in}\right) = 0 \tag{5.22}$$

and

$$V_\theta\left(r = R_{out}\right) = \Omega R_{out} \tag{5.23}$$

In this case the shear stress must satisfy[3]

$$\frac{d}{dr}\left(r^2 \tau_{r\theta}\right) = 0 \tag{5.24}$$

From this equation we can see that the shear stress depends on r and it is given by

$$\tau_{r\theta} = \mu\left[r\frac{d}{dr}\left(\frac{V_\theta}{r}\right)\right] = \frac{C}{r^2} \tag{5.25}$$

where C is a constant. Based on this we can write

$$\left[r\frac{d}{dr}\left(\frac{V_\theta}{r}\right)\right] = \frac{\tau_{r\theta}}{\mu} = \frac{C}{\mu r^2} \tag{5.26}$$

and by substituting this into Equation (5.19) we find

$$k\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{C^2}{\mu r^4} = 0 \tag{5.27}$$

Again we find a position and temperature dependent heat source whose position dependence strongly favors heat generation near the rod and replacing r everywhere by

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[3] From Bird, R.B., Stewart, W.E. and Lightfoot, E.N., 1960, Transport Phenomena, John Wiley & Sons, New York.

R_{in}, and then by R_{out}, we might imagine getting bounds on the critical condition. We might even anticipate that the position of the rod matters less here than before if R_{out} is large.

Our earlier work then is a model for some frictional heat generation problems, a better model the simpler the flow.

Now to get some idea of the effect of the geometry in frictional heating we return to plane Couette flow and consider a model of a honeycomb of channels. But first we deal with the simple case of unequal temperatures at the two plane walls.

5.2 Effect of Unequal Wall Temperatures in Frictional Heating

In Couette flow between parallel plane walls, the strength of the heat source for the cases of the same and different temperature at the walls is of interest. In the case of the same wall temperatures we have

$$k\frac{d^2T}{dx^2}+\frac{C^2}{\mu}=0 \tag{5.28}$$

where T must satisfy

$$T(x=L)=T_w \tag{5.29}$$

and

$$T(x=-L)=T_w \tag{5.30}$$

and where

$$\frac{dV_z}{dx}=\frac{\tau_{xz}}{\mu}=\frac{C}{\mu} \tag{5.31}$$

To start let μ_w/μ be a linear function of T (*i.e.*, $\mu_w/\mu(T)=1+s(T-T_w)$), and set $\dot{T}=T-T_w$. Based on Equation (5.28) $\dot{T}$ must satisfy

$$k\frac{d^2\dot{T}}{dx^2}+\frac{C^2}{\mu_w}\left(1+s\dot{T}\right)=0 \tag{5.32}$$

and Equation (5.31) will be

$$\frac{dV_z}{dx}=\frac{C}{\mu_w}\left(1+s\dot{T}\right) \tag{5.33}$$

Equations (5.32) and (5.33) can be rewritten in scaled form by defining $\dot{T}=\overline{T}\ T^*$, $V_z=\overline{V}_z\ V_z^*$ and $x=\overline{x}\ x^*$ where $\overline{T}=\frac{1}{s}$, $\overline{V}_z=\sqrt{k/\mu_w s}$ and $\overline{x}=L$. Then, the equations to be solved are

$$\frac{d^2T^*}{dx^{*2}}+\delta^2(1+T^*)=0\ ,\ \ -1<x^*<1 \tag{5.34}$$

and

$$\frac{dV_z^*}{dx^*}=\delta\left(1+T^*\right) \tag{5.35}$$

where

$$\delta^2=\frac{L^2sC^2}{k\mu_w}=\frac{L^2s\tau_{xz}^2}{k\mu_w}$$

The value of δ^2 indicates the strength of the heat source and the linear function $(1+T^*)$ indicates how fast it increases as T^* increases.

Notice that δ^2 can be zero for two reasons either s is zero, which only means μ is a constant, or C is zero which means τ_{xz} is zero, and no shear must mean no flow (*i.e.*, $V=0$). But in this problem we considered the case where μ depends linearly on T (*i.e.*, μ is not constant), then s cannot be zero.

Dropping the * 's, the steady state temperature rise must satisfy

$$\frac{d^2T}{dx^2}+\delta^2(1+T)=0 \tag{5.36}$$

where

$$T(x=1)=0 \tag{5.37}$$

and

$$T(x=-1)=0 \tag{5.38}$$

and Equation (5.35) will be

$$\frac{dV_z}{dx}=\delta\left(1+T\right) \tag{5.39}$$

The input variable of this problem (*i.e.*, Equation (5.36)), is δ^2 and T is the response and it depends on x.

Equation (5.36) can be solved to give

$$T=A\cos\delta x+B\sin\delta x-1\ , \quad -1<x<1 \tag{5.40}$$

where A and B satisfy

$$A\cos\delta+B\sin\delta-1=0\ , \ x=1 \tag{5.41}$$

and

$$A\cos\delta-B\sin\delta-1=0\ , \ x=-1 \tag{5.42}$$

whence T is given by

$$T=\frac{\cos\delta x}{\cos\delta}-1 \tag{5.43}$$

From Equation (5.43) the critical value of δ, the lowest value of δ where a solution fails to exist, is given by[4]

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[4] $\cos\delta=0$ for $\delta=n\pi/2$, where $n=1,3,5\ldots$, $\delta_{critical}$ will be the first of these values encountered in every δ from zero.

$$\delta_{critical} = \frac{\pi}{2} \tag{5.44}$$

Now, the speed of the moving boundary, V, can be determined by using Equations (5.39) and (5.43) which gives

$$dV_z = \delta\left(\frac{\cos\delta x}{\cos\delta}\right)dx \tag{5.45}$$

whereupon

$$V = \int_0^V dV_z = \delta\int_{-1}^{1}\left(\frac{\cos\delta x}{\cos\delta}\right)dx = 2\tan\delta \tag{5.46}$$

This is the speed of the wall as a function of δ and as δ ranges from zero to $\delta_{critical}$ the speed goes from zero to infinity. As the speed increases the stress at the wall increases, but as the temperature increases the rate at which the stress increases slows and the stress never exceeds a finite value no matter how high the speed (see Figure 5-4).

Now, consider the case with different temperatures on the two walls.[5] Let us replace $T(x=-1)=0$ by $T(x=-1)=\omega$, where $\omega = s(T_H - T_w) > 0$. Then, the steady state temperature rise is given by Equation (5.36), namely

$$\frac{d^2T}{dx^2} + \delta^2(1+T) = 0$$

where T must satisfy

$$T(x=1) = 0 \tag{5.47}$$

and

$$T(x=-1) = \omega \tag{5.48}$$

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[5] In this case μ_w will be measured at the cold wall *i.e.* $\mu_w = \mu(T_w)$.

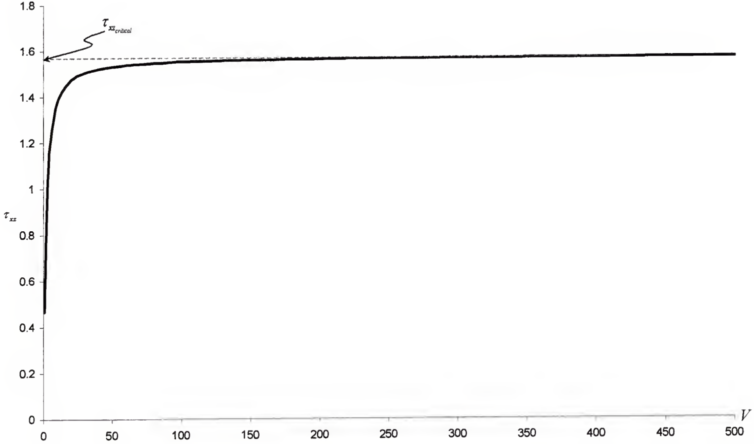

Figure 5-4. τ_{xz} *vs.* V .

The solution of this problem is given again by Equation (5.40) viz.,

$$T = A\cos\delta x + B\sin\delta x - 1\,,\quad -1 < x < 1$$

But the constants A and B now satisfy

$$A\cos\delta + B\sin\delta - 1 = 0\,,\ x = 1 \tag{5.49}$$

and

$$A\cos\delta - B\sin\delta - 1 = \omega\,,\ x = -1 \tag{5.50}$$

whence T for this problem turns out to be

$$T = \frac{(2+\omega)\cos\delta x\sin\delta - \omega\cos\delta\sin\delta x}{2\cos\delta\sin\delta} - 1 \tag{5.51}$$

Now to find the critical value of δ the denominator of the first term on the right-hand side of Equation (5.51) must vanish (*i.e.*, $2\cos\delta\sin\delta = 0$), where δ is the input variable and its value can be freely set.

Then, by using the *double angle formula* viz.,

$$\sin 2\delta = 2\cos\delta\sin\delta$$

Equation (5.51) becomes

$$T = \frac{(2+\omega)\cos\delta x\sin\delta - \omega\cos\delta\sin\delta x}{\sin 2\delta} - 1 \tag{5.52}$$

and hence the critical value of δ is given by[6]

$$\delta_{critical} = \frac{\pi}{2} \tag{5.53}$$

and this is just the critical value of δ obtained under symmetric conditions, given by Equation (5.44) and we can conclude that in *linear heat generation* the critical value of δ is independent of whether the temperatures on the boundaries are the same or not. This can be explained by studying the temperature variations in the two cases which will be presented later.

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[6] $\sin 2\delta = 0$ at $2\delta = n\pi$, where $n = 0,1,2,\ldots$.

Now, the speed of the moving boundary, V, can be obtained by using Equation (5.39), namely

$$dV_z = \delta\left(1+T\right)dx$$

where T for this case is given by Equation (5.52). Therefore, we have

$$dV_z = \delta\left(\frac{\left(2+\omega\right)\cos\delta x\sin\delta - \omega\cos\delta\sin\delta x}{\sin 2\delta}\right)dx \tag{5.54}$$

and carrying out the integration leads to

$$V = \int_0^V dV_z = \delta\int_{-1}^{1}\left(\frac{\left(2+\omega\right)\cos\delta x\sin\delta - \omega\cos\delta\sin\delta x}{\sin 2\delta}\right)dx = \frac{\left(2+\omega\right)\sin\delta}{\delta\cos\delta} \tag{5.55}$$

or

$$V = \frac{1}{\delta}\left(2+\omega\right)\tan\delta \tag{5.56}$$

and again V rises from zero to infinity as δ increases from zero to $\delta_{critical}$.

The temperature variations in the two cases are drawn in Figure 5-5. The curves are at the same δ and hence at the same shearing stress across the fluid. Near the hotter wall the viscosity is lower and hence the velocity gradient and the heat generation are higher than they otherwise would be. This explains the higher maximum and its shift toward the hotter wall. The hotter wall is still a sink albeit not as good a sink as before and the heat is generated closer to it and the heat generation is stronger. It would appear that the two effects just offset leaving $\delta_{critical}$ unchanged.

At least in this problem, the thermal stability of viscous heating does not depend on whether the two boundaries are at different temperatures or not. Even though the details

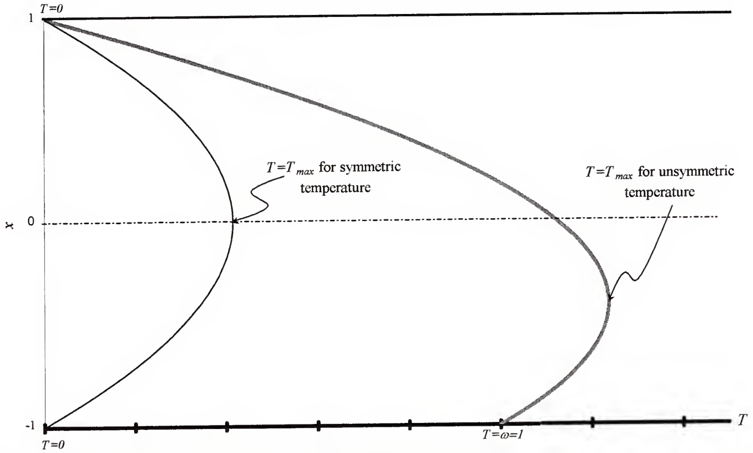

Figure 5-5. Temperature profile for Couette flow between parallel plane walls at $\delta = \pi/4$. (Symmetric and unsymmetric boundary conditions)

of the flow and the temperature will be affected by the temperature difference at the boundaries.

Now, let us see how a geometry change can affect the thermal stability of frictional heating problems. To begin we will first consider a simple problem. This is the problem of the thermal stability of viscous fluid flowing between parallel plane walls under the influence of a moving plane located in the center of the system. This is the subject of the following section.

5.3 Effect of Changing the Geometry on the Thermal Stability of Frictional Heating in Flow between Parallel Plane Walls Driven by the Motion of an Intermediate Plane

To begin, let us consider a Newtonian fluid confined in the space between two infinite parallel plane walls separated by a distance $2L$, as depicted in Figure 5-6. In the center (*i.e.*, at $x = 0$), a thin plane parallel to the two walls is set in motion. This plane divides the geometry into two equal parts, an upper region with $L > x > 0$ and a lower region with $0 > x > -L$ (*i.e.*, this is the symmetric case). The central plane moves at a velocity V in the z-direction while the two parallel plane walls are at rest. This causes the fluid in the two regions to flow in the z-direction (*i.e.*, $V_x = 0$), and because of this frictional heat is produced. This heat must be lost to the boundaries, which are held at a fixed temperature denoted T_w, if a steady state is to be achieved. Except for the geometry all is as it was before viz., linear fluidity, constant thermal conductivity, zero pressure gradient, etc.

Notice that this problem is just two Couette flow problems sharing a common moving boundary. Again the heat generated needs to be removed at the rate it is produced. This heat is removed by conduction to the two walls which serve as heat sinks.

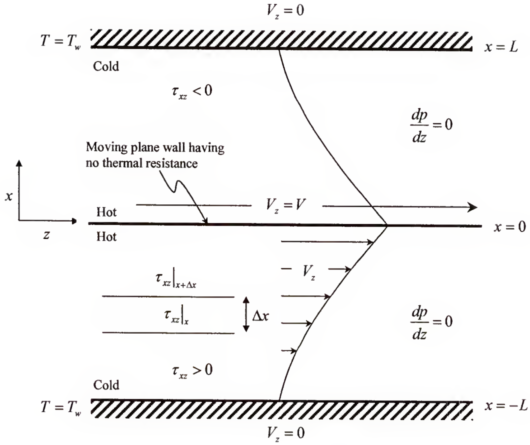

Figure 5-6. Couette flow between parallel plane walls due to the movement of a central plane on-center, $dp/dz = 0$. (Symmetric geometry)

In this on-center geometry the two Couette flows are the same and each is the same as the Couette flow considered earlier in a channel of width $2L$. Here each channel is of width L but there is only one sink, the central plane being adiabatic. Hence $\delta_{critical}$ is as it was before (*i.e.*, $\delta_{critical} = \pi/2$), and this would be the case even if the central plane offered some thermal resistance.

If now the central plane is shifted from its original position the geometry of this problem will change, being no longer symmetric about the moving plane. The effect of this change is depicted in Figure 5-7 where the upper region is thicker than the lower region (*i.e.*, $L^{**} > L^{*}$), but the overall thickness of the channel is still the same (*i.e.*, $L^{**} + L^{*} = 2L$). Again we have two Couette flows and if the moving plane were adiabatic the two flows would be independent, the upper region would be least stable and $\delta_{critical}$ would be[7] $\pi/2\hat{L}$. The temperature maximum would be at the moving plane, higher on the thicker side. Allowing the fluids to exchange heat would be expected to increase $\delta_{critical}$ but not to $\pi/2$ as the maximum temperature moves into the thicker region.

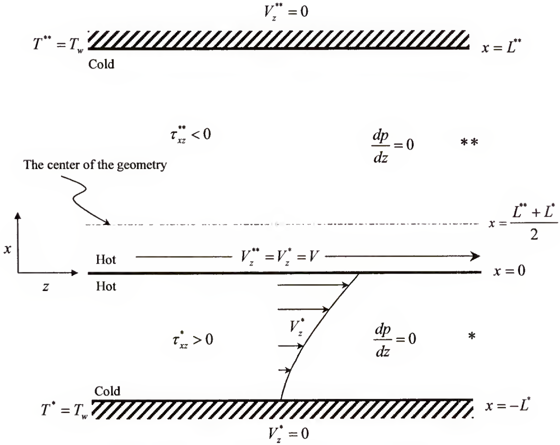

Figure 5-7. Couette flow between parallel plane walls due to the movement of a central plane off-center, $dp/dz = 0$. (Unsymmetric geometry)

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[7] $\hat{L}$ is the scaled width for the upper region *i.e.*, $\hat{L} = L^{**}/L$, where $\hat{L} > 1$.

Notice that τ_{xz} is not constant across the whole region (*i.e.*, $L^{**} > x > -L^{*}$), it is constant across each fluid, negative at the top (*i.e.*, $L^{**} > x > 0$), and positive at the bottom (*i.e.*, $0 > x > -L^{*}$), but τ_{xz}^{**} and τ_{xz}^{*} are not the same in absolute value.

Now, for the problem depicted in Figure 5-7 (*i.e.*, the unsymmetric problem), the steady state temperature rise, T, must satisfy

$$\frac{d^2T^{**}}{dx^2}+\left(\delta^{**}\right)^2(1+T^{**})=0 \quad , \quad 0<x<\hat{\hat{L}} \tag{5.57}$$

and

$$\frac{d^2T^{*}}{dx^2}+\left(\delta^{*}\right)^2(1+T^{*})=0 \quad , \quad -\hat{L}<x<0 \tag{5.58}$$

where $\hat{\hat{L}}$ and $\hat{L}$ are the scaled widths for the upper and the lower regions, respectively, which means

$$\hat{\hat{L}}=\frac{L^{**}}{L} \text{ and } \hat{L}=\frac{L^{*}}{L}$$

Some notation is illustrated in Figure 5-7. The boundary conditions to be satisfied are

$$T^{**}(x=\hat{\hat{L}})=0 \quad , \quad \text{at the upper wall} \tag{5.59}$$

$$T^{**}(x=0)=T^{*}(x=0) \quad , \quad \text{at the interior plane} \tag{5.60}$$

$$\frac{dT^{**}}{dx}(x=0)=\frac{dT^{*}}{dx}(x=0) \quad , \quad \text{at the interior plane} \tag{5.61}$$

and

$$T^{*}(x=-\hat{L})=0 \quad , \quad \text{at the lower wall} \tag{5.62}$$

and the speed of the flat plate can be obtained via Equation (5.39), which in this case can be written as

$$dV_z^{**} = -\delta^{**}\left(1+T^{**}\right)dx \tag{5.63}$$

where V_z^{**} goes down as x goes up and

$$dV_z^{*} = \delta^{*}\left(1+T^{*}\right)dx \tag{5.64}$$

where V_z^{*} goes up as x goes up and where

$$V_z^{**}\left(x=0\right) = V_z^{*}\left(x=0\right) = V \tag{5.65}$$

and

$$V_z^{**}\left(x=\hat{\hat{L}}\right) = V_z^{*}\left(x=-\hat{L}\right) = 0 \tag{5.66}$$

Then, to obtain the speed of the flat plate when the upper region is considered we have to integrate the left-hand side of Equation (5.63) with respect to V_z^{**} over $\left(V,0\right)$ and the right-hand side with respect to x over $\left(0,\hat{\hat{L}}\right)$ viz.,

$$V = \delta^{**}\int_0^{\hat{\hat{L}}}\left(1+T^{**}\right)dx \tag{5.67}$$

and to obtain the speed of the flat plate when the lower region is considered we must integrate the left-hand side of Equation (5.64) with respect to V_z^{*} over $\left(0,V\right)$ and the right-hand side with respect to x over $\left(-\hat{L},0\right)$ viz.,

$$V = \delta^{*}\int_{-\hat{L}}^{0}\left(1+T^{*}\right)dx \tag{5.68}$$

Then, from Equations (5.67) and (5.68) the following relationship must be true

$$\delta^{**}\int_0^{\hat{\hat{L}}}\left(1+T^{**}\right)dx = \delta^{*}\int_{-\hat{L}}^{0}\left(1+T^{*}\right)dx \tag{5.69}$$

The reason for this is that it is V that ties the two δ's together. Therefore, as inputs δ^{**} and δ^{*} are not independent, one is dependent on the other as the V calculation shows.

However, Equations (5.57) and (5.58) can be solved to give

$$T^{**} = A^{**}\cos\delta^{**}x + B^{**}\sin\delta^{**}x - 1 \tag{5.70}$$

and

$$T^{*} = A^{*}\cos\delta^{*}x + B^{*}\sin\delta^{*}x - 1 \tag{5.71}$$

Now we substitute these results into the four boundary conditions to determine A^{**}, B^{**}, A^{*} and B^{*}. This will give

$$A^{**}\cos\delta^{**}\hat{\hat{L}} + B\sin\delta^{**}\hat{\hat{L}} - 1 = 0 \ , \quad \text{at}\ x = \hat{\hat{L}} \tag{5.72}$$

$$A^{**} = A^{*} \ , \quad \text{at}\ x = 0 \tag{5.73}$$

$$\delta^{**}B^{**} = \delta^{*}B^{*} \ , \quad \text{at}\ x = 0 \tag{5.74}$$

and

$$A^{*}\cos\delta^{*}\hat{L} - B^{*}\sin\delta^{*}\hat{L} - 1 = 0 \ , \quad \text{at}\ x = -\hat{L} \tag{5.75}$$

Now by substituting Equations (5.73) and (5.74) into Equation (5.75) we get

$$A^{**}\cos\delta^{*}\hat{L} - \frac{\delta^{**}}{\delta^{*}}B^{**}\sin\delta^{*}\hat{L} - 1 = 0 \tag{5.76}$$

Then, the constants A^{**} and B^{**} can by determined by solving Equations (5.72) and (5.76) and, doing this, T^{**} will turn out to be

$$\begin{aligned} T^{**} = \Bigg[& \left(\cos\delta^{**}x - \cos\delta^{**}\hat{\hat{L}}\right)\varphi \\ & + \cos\delta^{**}\hat{\hat{L}}\left(\cos\delta^{*}\hat{L} - \cos\delta^{**}\hat{\hat{L}}\right)\left(\sin\delta^{**}x - \frac{\sin\delta^{**}\hat{\hat{L}}\cos\delta^{**}x}{\cos\delta^{**}\hat{\hat{L}}}\right)\Bigg] \Big/ \varphi\cos\delta^{**}\hat{\hat{L}} \end{aligned} \tag{5.77}$$

where T^{*} will be

$$T^{*}=\left[\left(\cos\delta^{*}x-\cos\delta^{**}\hat{\hat{L}}\right)\varphi\right.$$
$$\left.+\cos\delta^{**}\hat{\hat{L}}\left(\cos\delta^{*}\hat{L}-\cos\delta^{**}\hat{\hat{L}}\right)\left(\frac{\delta^{**}}{\delta^{*}}\sin\delta^{*}x-\frac{\sin\delta^{**}\hat{\hat{L}}\cos\delta^{*}x}{\cos\delta^{**}\hat{\hat{L}}}\right)\right]\Big/\varphi\cos\delta^{**}\hat{\hat{L}} \tag{5.78}$$

where

$$\varphi=\left(\sin\delta^{**}\hat{\hat{L}}\cos\delta^{*}\hat{L}+\frac{\delta^{**}}{\delta^{*}}\sin\delta^{*}\hat{L}\cos\delta^{**}\hat{\hat{L}}\right) \tag{5.79}$$

The aim of this calculation is to find the critical condition (the condition beyond which no solution exists *i.e.*, where T^{**} and T^{*} cannot be found). From Equations (5.77) and (5.78) one can see that T^{**} and T^{*} are not defined when $\varphi\cos\delta^{**}\hat{\hat{L}}=0$. To find the critical values of δ^{**} and δ^{*} at which T^{**} and T^{*} cease to exist we must have either $\cos\delta^{**}\hat{\hat{L}}=0$ or $\varphi=0$. If the critical condition comes from $\cos\delta^{**}\hat{\hat{L}}=0$ the critical value of δ^{**} turns out to be

$$\delta^{**}_{critical}=\frac{\pi}{2\hat{\hat{L}}} \tag{5.80}$$

which is what it was when the moving plane were adiabatic. But this result cannot be true because, as we explained earlier, the critical value of δ for the upper region, $\delta^{**}_{critical}$, in this case must be higher than it is when the moving plane were adiabatic. Therefore, the only possibility is $\varphi=0$, which means

$$\sin\delta^{**}\hat{\hat{L}}\cos\delta^{*}\hat{L}=-\frac{\delta^{**}}{\delta^{*}}\sin\delta^{*}\hat{L}\cos\delta^{**}\hat{\hat{L}} \tag{5.81}$$

which can be rearranged and written as

$$\delta^{*} = -\delta^{**} \frac{\tan \delta^{*} \hat{L}}{\tan \delta^{**} \hat{\hat{L}}} \tag{5.82}$$

This does not determine the critical values of δ^{**} and δ^{*} until this dependence in one another is established. Now, as we found earlier in Equation (5.69), δ^{**} and δ^{*} are tied together by the common value of V whence

$$\delta^{**} \int_{0}^{\hat{\hat{L}}} \left(1 + T^{**}\right) dx = \delta^{*} \int_{-\hat{L}}^{0} \left(1 + T^{*}\right) dx$$

where the integrations can be carried out by using Equations (5.77) and (5.78). But in all the problems done earlier we know that V is infinite at critical which corresponds to T^{**} and T^{*} both being infinite at critical. Because of this, it becomes hard to solve Equations (5.69) and (5.82) simultaneously. Instead calculate δ^{**} and δ^{*} from Equations (5.67) and (5.68) viz.,

$$V = \delta^{**} \int_{0}^{\hat{\hat{L}}} \left(1 + T^{**}\right) dx = f\left(\delta^{**}, \delta^{*}\right)$$

and

$$V = \delta^{*} \int_{-\hat{L}}^{0} \left(1 + T^{*}\right) dx = g\left(\delta^{**}, \delta^{*}\right)$$

The plan is to calculate δ^{**} and δ^{*} each for an increasing sequence of V 's until δ^{**} and δ^{*} become constant. These constant values are the critical values of δ^{**} and δ^{*} (*i.e.*, $\delta^{**}_{critical}$ and $\delta^{*}_{critical}$). This is what happened earlier in simpler problems and there is no reason to think it will not be true here. In the following we are going to explain how to go about doing this.

Now for given values of $\hat{\hat{L}}$ and $\hat{L}$, let us set a value for V in Equation (5.67) and then put δ^{**}, where δ^{**} and δ^{*} can have any value between zero and infinity, and seek value of δ^{*} that satisfy Equation (5.67). After that put another δ^{**} and again seek values of δ^{*} that satisfy Equation (5.67) and so on. The next step is to plot δ^{**} *vs.* δ^{*}. Then set the same value of V in Equation (5.68) and repeat what we did for Equation (5.67) (*i.e.*, put δ^{**} and seek value of δ^{*} that satisfy Equation (5.68)), then plot δ^{**} *vs.* δ^{*}. Doing this we get two graphs for δ^{**} *vs.* δ^{*}, one from each of the two equations. Figure 5-8 depicts such a plot of δ^{**} *vs.* δ^{*} at a given $\hat{\hat{L}}$, $\hat{L}$ and V. The intersection points between the two graphs indicates that the two equations are equal and this satisfies the relationship given by Equation (5.69), namely

$$\delta^{**}\int_{0}^{\hat{\hat{L}}}\left(1+T^{**}\right)dx=\delta^{*}\int_{-\hat{L}}^{0}\left(1+T^{*}\right)dx$$

Therefore, for the given values of $\hat{\hat{L}}$, $\hat{L}$ and V the values of δ^{**} and δ^{*} become known. Notice that the first intersection point, the one with lowest values of δ^{**} and δ^{*}, will be chosen because it is the first one to be hit. Then let us set another value for V and carry on the same steps and record the values of δ^{**} and δ^{*} at which Equation (5.69) will be satisfied, and then repeat the same thing for another value of V and so on. This process must continue until the values of δ^{**} and δ^{*} become fixed as V goes to infinity. These fixed values of δ^{**} and δ^{*} are equal to $\delta^{**}_{critical}$ and $\delta^{*}_{critical}$. This can be checked by substituting the obtained $\delta^{**}_{critical}$ and $\delta^{*}_{critical}$ into Equation (5.82) and see if the two sides become equal. Some results are illustrated in Table 5-1.

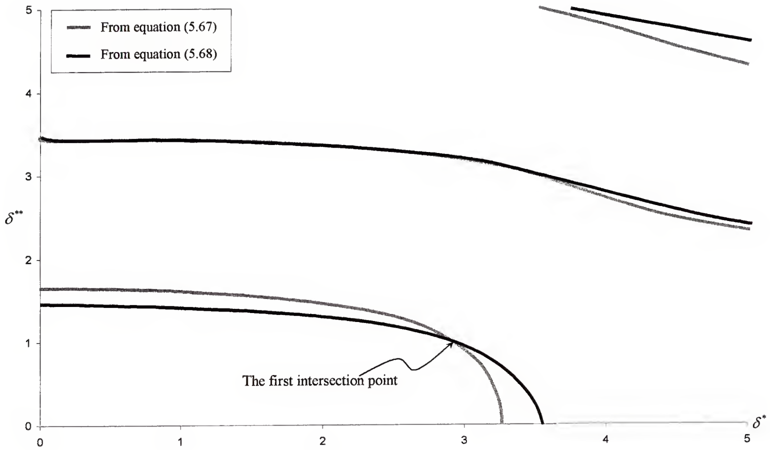

Figure 5-8. δ^{**} *vs.* δ^{*} for $\hat{\hat{L}} = 1.5$, $\hat{L} = 0.5$ and $V = 10$.

Table 5-1. Values of δ^{**} and δ^{*} for $\hat{\hat{L}} = 1.5$, $\hat{L} = 0.5$ and increasing values of V.

V	δ^{**}	δ^{*}	V	δ^{**}	δ^{*}
1	0.525	1.580	40	1.040	3.090
2	0.720	2.230	50	1.050	3.110
3	0.830	2.500	60	1.055	3.120
4	0.880	2.660	70	1.060	3.128
5	0.920	2.750	80	1.060	3.128
10	0.980	2.940	90	1.060	3.128
20	1.020	3.040	100	1.060	3.128
30	1.030	3.080	∞	1.060	3.128

Table 5-1 shows that the values of δ^{**} and δ^{*} become fixed at $V = 70$. Therefore, at $\hat{\hat{L}} = 1.5$ and $\hat{L} = 0.5$ we get $\delta^{**}_{critical} = 1.060$ and $\delta^{*}_{critical} = 3.128$. These results satisfy Equation (5.82). Moreover, the obtained value of $\delta^{**}_{critical}$ agrees with our earlier expectation about how $\delta^{**}_{critical}$ for this case must lie between $\delta^{**}_{critical}$ for an off-center, adiabatic, moving plane and $\delta_{critical}$ for an on-center moving plane (*i.e.*, $\pi/2\hat{\hat{L}} < \delta^{**}_{critical} < \pi/2$), where at the given value of $\hat{\hat{L}}$ we find that $\pi/2\hat{\hat{L}} = 1.047$. Therefore, the case of an off-center, adiabatic, moving plane is the most dangerous because it has the lowest $\delta^{**}_{critical}$ while the case of an on-center moving plane is the less dangerous. From this we can conclude that the thermal stability of viscous fluid flow problems can be affected by change in the geometry of the system.

The temperature and the velocity profiles for this case are depicted in Figure 5-9 and Figure 5-10, respectively, at $V = 10$. Observe that the maximum temperature lies in the thicker region as we expected earlier.

These simple planar problems serving as an introduction can be solved no matter the spacing. The reason they are solvable is that V_z can be eliminated from our problem

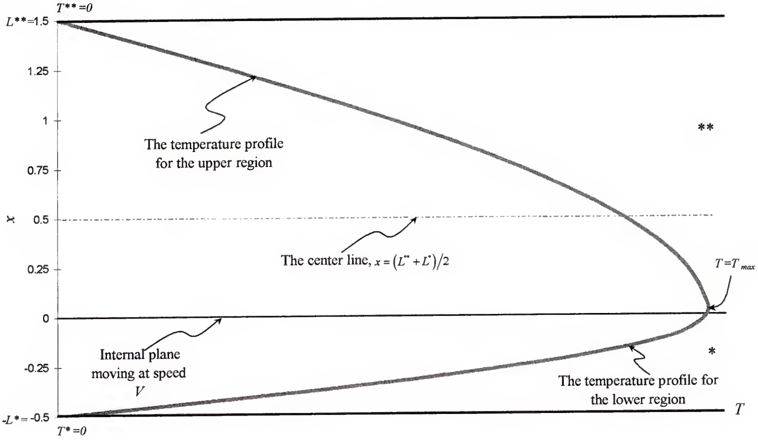

Figure 5-9. Temperature profile for Couette flow between parallel plane walls due to the motion of an off-center plane, $V = 10$. (Unsymmetric geometry)

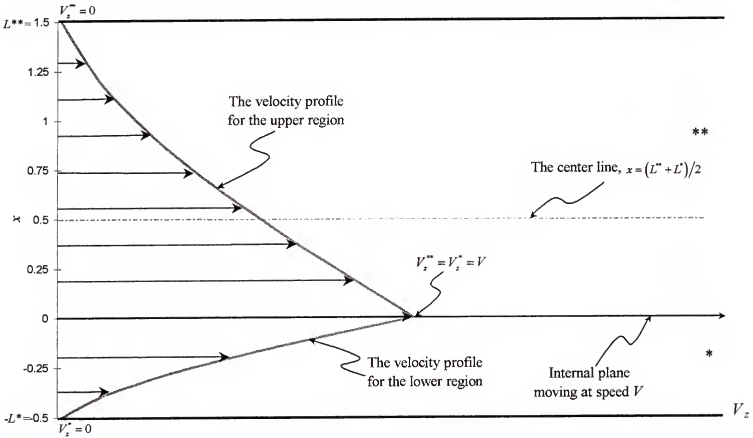

Figure 5-10. Velocity profile for Couette flow between parallel plane walls due to the motion of an off-center plane, $V=10$.
(Unsymmetric geometry)

in V_z and T reducing it to a problem in T alone. This never changes as the spacing changes and this is due to the fact that the heat generated depends only on one shearing stress, τ_{xz}, no matter the spacing. In cylindrical problems this advantage is lost as we shall soon see.

5.4 Thermal Stability of Frictional Heating in Axial Flow Between a Rod and a Pipe due to Viscous Heating

This section begins a study of frictional heating in the case where a fluid is made to flow axially in the annular region between a rod and a pipe. The flow is due to the rod moving relative to the pipe. The base case has the rod on-center. The displaced case has the rod off-center. On displacement the flow remains parallel to the centerline of the pipe but its variations becomes more complicated than heretofore and we lose the simplification that only one shearing stress enters the problem. The fluid in the annular space carried along by the rod generates frictional heat, which must be conducted to the pipe wall. Figure 5-11 depicts the problem. Most of the assumptions remain as before. Only the geometry has changed, but even the notation is not new. The radii of the rod and the pipe are denoted R_{in} and R_{out}, respectively. The rod is displaced a distance ε off-center. The pipe wall is the heat sink and the rod wall is adiabatic. The rod moves at a velocity V in the *z*-direction while the pipe is at rest. Hence, the fluid will flow in the *z*-direction due to the motion of the rod and V_r and V_θ both remain zero.

In the symmetric case (Figure 5-11A), the heat is conducted radially outward from the adiabatic rod to the pipe wall.[8] Then, a shift in the inner cylinder, as it is shown in Figure 5-11B, introduces additional heat transfer by conduction in the azimuthal direction but it also creates both a thick and a thin heat- producing region. It is the thick region

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[8] A more practical, and more easier, case is to have a cooling rod inside the pipe.

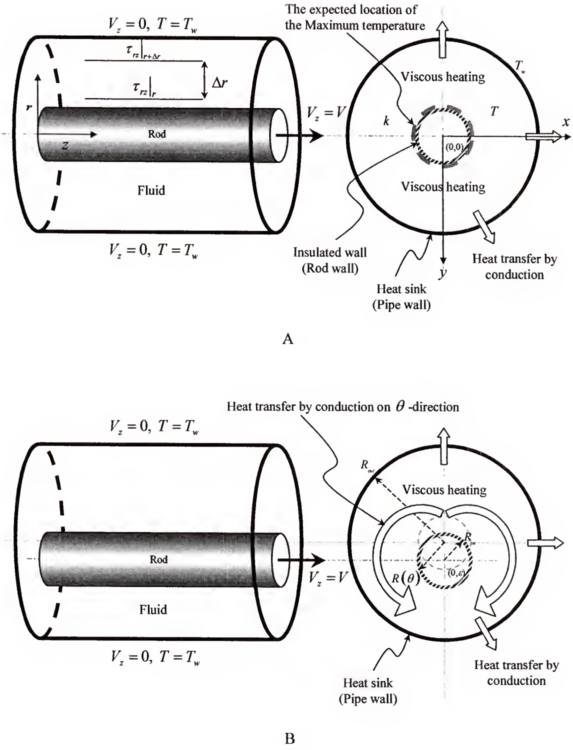

Figure 5-11. Solid rod moving at speed V inside a pipe. A) On-center. B) Off-center

that is dangerous and if the azimuthal conduction is not enough to meet the increased demand for heat removal, a displaced rod will be more dangerous than an on-center rod. To find out whether off-center or on-center rods are more dangerous requires a calculation.

Let us begin with the on-center case, where the temperature rise is described by

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{1}{k\mu}\tau_{rz}^{2}=0 \tag{5.83}$$

where

$$\tau_{rz}=\mu\frac{dV_z}{dr}$$

and where the boundary conditions to be satisfied are

$$T\left(r=R_{out}\right)=T_w \tag{5.84}$$

$$\frac{dT}{dr}\left(r=R_{in}\right)=0 \tag{5.85}$$

$$V_z\left(r=R_{out}\right)=0 \tag{5.86}$$

and

$$V_z\left(r=R_{in}\right)=V \tag{5.87}$$

Now τ_{rz} satisfies

$$\frac{d\left(r\tau_{rz}\right)}{dr}=0 \tag{5.88}$$

whence τ_{rz} is given by

$$\tau_{rz}=\frac{C}{r} \tag{5.89}$$

where C is a constant, $C=R_{in}\,\tau_{rz}\big|_{R_{in}}$. Based on this the equation for T is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{C^2}{k\mu r^2}=0 \tag{5.90}$$

Introducing $T^* = s\left(T - T_w\right)$ and $\mu_w/\mu(T) = 1 + s(T - T_w)$ we must solve

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT^*}{dr}\right)+\frac{\delta^2}{r^2}\left(1+T^*\right)=0 \tag{5.91}$$

where

$$\delta^2 = \frac{sC^2}{k\mu_w} = \frac{sR_{in}^2\,\tau_{rz}^2\big|_{R_{in}}}{k\mu_w}$$

and on dropping the * 's we have

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\delta^2}{r^2}\left(1+T\right)=0 \tag{5.92}$$

where

$$T\left(r = R_{out}\right) = 0 \tag{5.93}$$

and

$$\frac{dT}{dr}\left(r = R_{in}\right) = 0 \tag{5.94}$$

Equation (5.92) can be arranged in the form of a Cauchy-Euler equation viz.,

$$r^2\frac{d^2T}{dr^2}+r\frac{dT}{dr}+\delta^2 T = -\delta^2 \tag{5.95}$$

which can be solved by substituting $r = e^{\xi}$ to get

$$\frac{d^2T}{d\xi^2}+\delta^2 T = -\delta^2 \tag{5.96}$$

Hence T is given by

$$T = A\cos\left(\delta\xi\right)+B\sin\left(\delta\xi\right)-1 \tag{5.97}$$

where A and B are constants, which in terms of r is

$$T = A\cos(\delta\ln(r)) + B\sin(\delta\ln(r)) - 1 \tag{5.98}$$

To find A and B substitute this formula for T into Equations (5.93) and (5.94) to get

$$A\cos(\delta\ln(R_{out})) + B\sin(\delta\ln(R_{out})) - 1 = 0 \tag{5.99}$$

and

$$-A\sin(\delta\ln(R_{in})) + B\cos(\delta\ln(R_{in})) = 0 \tag{5.100}$$

whence T for this problem can be written as

$$T = \frac{\cos\left(\delta\ln\left(\frac{r}{R_{in}}\right)\right)}{\cos\left(\delta\ln\left(\frac{R_{out}}{R_{in}}\right)\right)} - 1 \tag{5.101}$$

The smallest value of δ at which T is not defined is its critical value, and in this on-centered case it is

$$\delta_{critical} = \frac{\pi}{2\ln\left(\frac{R_{out}}{R_{in}}\right)} \tag{5.102}$$

For any value of δ the corresponding speed of the rod, V, can be obtained from

$$\frac{dV_z}{dr} = \frac{\tau_{rz}}{\mu} = \frac{C}{\mu r} = \frac{1}{r}\frac{C}{\mu_w}(1+T) \tag{5.103}$$

where $\tau_{rz} = C/r$ and C is negative. But we know that

$$\delta^2 = \frac{sC^2}{k\mu_w}$$

and δ has been taken to be positive whence

$$C = \frac{-\delta}{\sqrt{\dfrac{s}{k\mu_w}}} \tag{5.104}$$

whereupon scaling V_z by $\sqrt{k/s\mu_w}$ and dropping the *'s we have

$$dV_z = -\delta\frac{(1+T)}{r}dr \tag{5.105}$$

where T is given by Equation (5.101). Therefore, Equation (5.105) can be written as

$$dV_z = -\left(\frac{\cos\left(\delta\ln\left(\dfrac{r}{R_{in}}\right)\right)}{\cos\left(\delta\ln\left(\dfrac{R_{out}}{R_{in}}\right)\right)}\right)\delta d\ln\left(\frac{r}{R_{in}}\right) \tag{5.106}$$

and on carrying out the integrations we get

$$V = \tan\left(\delta\ln\left(\frac{R_{out}}{R_{in}}\right)\right) \tag{5.107}$$

Now, before turning to the off-center case, we might anticipate a major difficulty. All the problems in this chapter, so far, have been made to look like the simpler problems treated in the earlier chapters because, while they introduce both T and V_z as unknowns, we have been able to eliminate V_z in favor of T due to the fact that the stress can be determined before either T or V_z. Once we move the rod this will no longer be true because of the fact that both τ_{rz} and $\tau_{\theta z}$ determine the heat generation but neither can be found a priori. We will set this calculation up but we cannot carry it out without some approximations, one such would be to not perturb the flow and another would be to hold the viscosity fixed and break the positive feedback. We will deal with this case because it is somewhat interesting.

Now, turn to the problem presented in Figure 5-11B where the temperature rise is described by the thermal energy equation

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}\right]+\frac{1}{k\mu}\left[\tau_{rz}^2+\tau_{\theta z}^2\right]=0 \tag{5.108}$$

while the velocity of the fluid must satisfy

$$\frac{1}{r}\frac{\partial(r\tau_{rz})}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta}=0 \tag{5.109}$$

where

$$\tau_{rz}=\mu\frac{dV_z}{dr} \tag{5.110}$$

and

$$\tau_{\theta z}=\frac{\mu}{r}\frac{dV_z}{d\theta} \tag{5.111}$$

Now Equation (5.109) does not help us simplify Equation (5.108) and hence it would appear that eliminating τ_{rz} and $\tau_{\theta z}$ in favor of V_z might be the thing to do, even though this will not lead us to a variable such as δ which appears in all the foregoing problems. Doing this we have

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}\right]+\frac{\mu}{k}\left[\left(\frac{\partial V_z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial V_z}{\partial \theta}\right)^2\right]=0 \tag{5.112}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu\frac{\partial V_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\mu\frac{\partial V_z}{\partial\theta}\right)=0 \tag{5.113}$$

where the boundary conditions to be satisfied are

$$T(r=R_{out})=T_w \tag{5.114}$$

$$\vec{n}\cdot\nabla T\left(r=R\left(\theta,\varepsilon\right)\right)=0 \tag{5.115}$$

$$V_z\left(r=R_{out}\right)=0 \tag{5.116}$$

and

$$V_z\left(r=R\left(\theta,\varepsilon\right)\right)=V \tag{5.117}$$

Now, to solve Equation (5.112), we must find V_z. To do this we turn to Equation (5.113), which is an equation in V_z but it also contains μ which depends on T via $\mu_w/\mu(T)=1+s(T-T_w)$ and T is an unknown that cannot be found unless V_z is found. Hence Equations (5.112) and (5.113) must be solved simultaneously. However, the base, or on-center solution is known and before we give up we might look at the perturbation equations. The details for these equations are familiar from earlier chapters and we give these equations in their final form. At order ε^o, which is the on-centered problem having no θ variation, the domain equations are

$$\frac{1}{r_o}\frac{d}{dr_o}\left(r_o\frac{dT_o}{dr_o}\right)+\frac{\mu_o}{k}\left(\frac{dV_{z_o}}{dr_o}\right)^2=0 \tag{5.118}$$

and

$$\frac{1}{r_o}\frac{d}{dr_o}\left(r_o\mu_o\frac{dV_{z_o}}{dr_o}\right)=0 \tag{5.119}$$

where

$$\mu_o=f\left(T_o\right) \tag{5.120}$$

and where at the boundaries T_o and V_{z_o} must satisfy

$$T_o(r_o=R_{out})=0 \tag{5.121}$$

$$\frac{dT_o}{dr_o}(r_o = R_{in}) = 0 \tag{5.122}$$

$$V_{z_o}\left(r_o = R_{out}\right) = 0 \tag{5.123}$$

and

$$V_{z_o}\left(r_o = R_{in}\right) = V \tag{5.124}$$

At order ε^1 the equations on the reference domain are

$$\left[\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o\frac{\partial T_1}{\partial r_o}\right) + \frac{1}{r_o^2}\frac{\partial^2 T_1}{\partial \theta_o^2}\right] + 2\frac{\mu_o}{k}\frac{\partial V_{z_o}}{\partial r_o}\frac{\partial V_{z_1}}{\partial r_o} + \frac{\mu_1}{k}\left(\frac{\partial V_{z_o}}{\partial r_o}\right)^2 = 0 \tag{5.125}$$

and

$$\left[\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o\mu_o\frac{\partial V_{z_1}}{\partial r_o} + r_o\mu_1\frac{\partial V_{z_o}}{\partial r_o}\right) + \frac{1}{r_o^2}\frac{\partial}{\partial \theta_o}\left(\mu_o\frac{\partial V_{z_1}}{\partial \theta_o} + \mu_1\frac{\partial V_{z_o}}{\partial \theta_o}\right)\right] = 0 \tag{5.126}$$

where

$$\mu_1 = T_1\frac{df\left(T_o\right)}{dr_o} \tag{5.127}$$

and where at the boundaries T_1 and V_{z_1} must satisfy

$$T_1(r_o = R_{out}) = 0 \tag{5.128}$$

$$\frac{dT_1}{dr_o}(r_o = R_{in}) = -R_1\left.\frac{d^2T_o}{dr_o^2}\right|_{r_o=R_{in}} \tag{5.129}$$

$$V_{z_1}\left(r_o = R_{out}\right) = 0 \tag{5.130}$$

and

$$V_{z_1}\left(r_o = R_{in}\right) = -R_1\left.\frac{dV_{z_o}}{dr_o}\right|_{r_o=R_{in}} \tag{5.131}$$

At order ε^2, the domain equations are

$$\left[\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o\frac{\partial T_2}{\partial r_o}\right)+\frac{1}{r_o^2}\frac{\partial^2 T_2}{\partial \theta_o^2}\right] + 2\frac{\mu_o}{k}\left[\left(\frac{\partial V_{z_1}}{\partial r_o}\right)^2+\frac{\partial V_{z_o}}{\partial r_o}\frac{\partial V_{z_2}}{\partial r_o}+\frac{1}{r_o^2}\left(\frac{\partial V_{z_1}}{\partial \theta_o}\right)^2\right] + 2\frac{\mu_1}{k}\frac{\partial V_{z_o}}{\partial r_o}\frac{\partial V_{z_1}}{\partial r_o}+\frac{\mu_2}{2k}\left(\frac{\partial V_{z_o}}{\partial r_o}\right)^2=0 \tag{5.132}$$

and

$$\left[\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o\frac{\mu_o}{2}\frac{\partial V_{z_2}}{\partial r_o}+r_o\mu_1\frac{\partial V_{z_1}}{\partial r_o}+r_o\frac{\mu_2}{2}\frac{\partial V_{z_o}}{\partial r_o}\right) + \frac{1}{r_o^2}\frac{\partial}{\partial \theta_o}\left(\frac{\mu_o}{2}\frac{\partial V_{z_2}}{\partial \theta_o}+\mu_1\frac{\partial V_{z_1}}{\partial \theta_o}+\frac{\mu_2}{2}\frac{\partial V_{z_o}}{\partial \theta_o}\right)\right]=0 \tag{5.133}$$

where

$$\mu_2=\frac{1}{2}\left(T_2\frac{df(T_o)}{dr_o}+T_1^2\frac{d^2f(T_o)}{dr_o^2}\right) \tag{5.134}$$

and where at the boundaries T_2 and V_{z_2} must satisfy

$$T_2(r_o=R_{out})=0 \tag{5.135}$$

$$\frac{\partial T_2}{\partial r_o}(r_o=R_{in})=-2R_1\left.\frac{\partial^2 T_1}{\partial r_o^2}\right|_{r_o=R_{in}}-R_1^2\left.\frac{\partial^3 T_o}{\partial r_o^3}\right|_{r_o=R_{in}} - R_2\left.\frac{\partial^2 T_o}{\partial r_o^2}\right|_{r_o=R_{in}}-2\left.\frac{\sin\theta}{R_{in}^2}\frac{\partial T_1}{\partial \theta}\right|_{r_o=R_{in}} \tag{5.136}$$

$$V_{z_2}(r_o=R_{out})=0 \tag{5.137}$$

and

$$V_{z_2}(r_o=R_{in})=-2R_1\left.\frac{dV_{z_1}}{dr_o}\right|_{r_o=R_{in}}-R_1^2\left.\frac{d^2V_{z_o}}{dr_o^2}\right|_{r_o=R_{in}}-R_2\left.\frac{dV_{z_o}}{dr_o}\right|_{r_o=R_{in}} \tag{5.138}$$

Now, as we explained earlier, there is some interest in solving this problem at constant viscosity. Then Equation (5.113), can be written as

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{\partial V_z}{\partial \theta}\right)\right]=0 \qquad (5.139)$$

which can be solved to obtain V_z. This breaks the coupling just as did conditions such as $\tau_{rz} = \text{constant}$ but it results in a less interesting problem. Nevertheless this will be presented in the following.

Let us start with the on-centered case where the temperature rise can be described by Equation (5.90), namely

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{C^2}{k\mu r^2}=0$$

and where the boundary conditions to be satisfied are given by Equations (5.84) and (5.85).

In terms of $T^* = T - T_w$ our problem; on dropping the *'s, is

$$r^2\frac{d^2T}{dr^2}+r\frac{dT}{dr}+\delta^2=0 \qquad (5.140)$$

where

$$T\left(r=R_{out}\right)=0 \qquad (5.141)$$

and

$$\frac{dT}{dr}\left(r=R_{in}\right)=0 \qquad (5.142)$$

and where

$$\delta^2=\frac{C^2}{k\mu T_w}$$

In terms of ξ where $r = e^{\xi}$, we get

$$\frac{d^2T}{d\xi^2} = -\delta^2 \tag{5.143}$$

whence

$$T = A\xi + B - \frac{\delta^2}{2}\xi^2 \tag{5.144}$$

where A and B are constants to be determined and putting $\xi = \ln(r)$ into Equation (5.144) we get

$$T = A\ln(r) + B - \frac{\delta^2}{2}(\ln(r))^2 \tag{5.145}$$

To find A and B substitute this formula for T into Equations (5.141) and (5.142) to get

$$A\ln(R_{out}) + B - \frac{\delta^2}{2}(\ln(R_{out}))^2 = 0 \tag{5.146}$$

and

$$A - \delta^2\ln(R_{in}) = 0 \tag{5.147}$$

whence T can be written as

$$T = \frac{\delta^2}{2}\left[(\ln(R_{out}))^2 - (\ln(r))^2 + 2\ln(R_{in})\ln\left(\frac{r}{R_{out}}\right)\right] \tag{5.148}$$

Of course there is no critical value of δ. The solution exists for all finite values of δ.

The speed of the rod, V, can be obtained from

$$\frac{dV_z}{dr} = \frac{1}{\mu}\tau_{rz} = \frac{1}{\mu}\frac{C}{r} \tag{5.149}$$

whereupon scaling V_z by $\sqrt{kT_w/\mu}$ and on dropping the *'s we have

$$dV_z = -\delta \frac{1}{r} dr \tag{5.150}$$

and on carrying out the integrations we get

$$V = \delta \ln\left(\frac{R_{out}}{R_{in}}\right) \tag{5.151}$$

We may notice that the shear stress, τ_{rz}, is given by C/r whether or not the viscosity is temperature dependent.

Now, when the rod is off-center, we have

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}\right] + \frac{\mu}{k}\left[\left(\frac{\partial V_z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial V_z}{\partial \theta}\right)^2\right] = 0$$

and

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{\partial V_z}{\partial \theta}\right)\right] = 0$$

whence

$$T\left(r = R_{out}\right) = 0$$

$$\vec{n} \cdot \nabla T\left(r = R(\theta, \varepsilon)\right) = 0$$

$$V_z\left(r = R_{out}\right) = 0$$

and

$$V_z\left(r = R(\theta, \varepsilon)\right) = V$$

where T is the temperature above T_w.

The rod is displaced from the center of the pipe by an amount ε. The case $\varepsilon = 0$ is the case solved above and our aim is to calculate corrections to orders one and two. Our first job is to present the problems in $T_o, T_1, T_2, \ldots$ and in $V_{z_o}, V_{z_1}, V_{z_2}, \ldots$ on the reference domain (*i.e.* to perturb the equations). The details are familiar from earlier chapters and we give these equations in their final form. Thus, the equations to be solved with their appropriate boundary conditions[9] are presented as follows: First at order ε^o, which is the on-centered problem having no θ variation, the domain equations are

$$\frac{1}{r_o}\frac{d}{dr_o}\left(r_o\frac{dT_o}{dr_o}\right)+\frac{\mu}{k}\left(\frac{dV_{z_o}}{dr_o}\right)^2=0 \tag{5.152}$$

and

$$\frac{1}{r_o}\frac{d}{dr_o}\left(r_o\frac{dV_{z_o}}{dr_o}\right)=0 \tag{5.153}$$

where

$$T_o(r_o=R_{out})=0 \tag{5.154}$$

$$\frac{dT_o}{dr_o}(r_o=R_{in})=0 \tag{5.155}$$

$$V_{z_o}\left(r_o=R_{out}\right)=0 \tag{5.156}$$

and

$$V_{z_o}\left(r_o=R_{in}\right)=V \tag{5.157}$$

At order ε^1 the equations on the reference domain are

$$\left[\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o\frac{\partial T_1}{\partial r_o}\right)+\frac{1}{r_o^2}\frac{\partial^2 T_1}{\partial\theta_o^2}\right]+2\frac{\mu}{k}\frac{\partial V_{z_o}}{\partial r_o}\frac{\partial V_{z_1}}{\partial r_o}=0 \tag{5.158}$$

---

[9] Note that the perturbed temperature boundary condition is no longer adiabatic because it tries to simulate adiabatic conditions at the displaced surface by putting a condition other than adiabatic on the reference surface. Likeness the perturbed velocity boundary condition is no longer V because it tries to simulate V by putting a condition other than V on the reference surface.

and

$$\left[\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o\frac{\partial V_{z_1}}{\partial r_o}\right)+\frac{1}{r_o^2}\frac{\partial}{\partial \theta_o}\left(\frac{\partial V_{z_1}}{\partial \theta_o}\right)\right]=0 \tag{5.159}$$

where at the boundaries T_1 and V_{z_1} must satisfy

$$T_1(r_o=R_{out})=0 \tag{5.160}$$

$$\frac{\partial T_1}{\partial r_o}(r_o=R_{in})=-R_1\left.\frac{\partial^2 T_o}{\partial r_o^2}\right|_{r_o=R_{in}} \tag{5.161}$$

$$V_{z_1}\left(r_o=R_{out}\right)=0 \tag{5.162}$$

and

$$V_{z_1}\left(r_o=R_{in}\right)=-R_1\left.\frac{dV_{z_o}}{dr_o}\right|_{r_o=R_{in}} \tag{5.163}$$

At order ε^2, the domain equations are

$$\left[\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o\frac{\partial T_2}{\partial r_o}\right)+\frac{1}{r_o^2}\frac{\partial^2 T_2}{\partial \theta_o^2}\right]+2\frac{\mu}{k}\left[\left(\frac{\partial V_{z_1}}{\partial r_o}\right)^2+\frac{\partial V_{z_o}}{\partial r_o}\frac{\partial V_{z_2}}{\partial r_o}+\frac{1}{r_o^2}\left(\frac{\partial V_{z_1}}{\partial \theta_o}\right)^2\right]=0 \tag{5.164}$$

and

$$\left[\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o\frac{\partial V_{z_2}}{\partial r_o}\right)+\frac{1}{r_o^2}\frac{\partial}{\partial \theta_o}\left(\frac{\partial V_{z_2}}{\partial \theta_o}\right)\right]=0 \tag{5.165}$$

where

$$T_2(r_o=R_{out})=0 \tag{5.166}$$

$$\begin{aligned}\frac{\partial T_2}{\partial r_o}(r_o=R_{in})=&-2R_1\left.\frac{\partial^2 T_1}{\partial r_o^2}\right|_{r_o=R_{in}}-R_1^2\left.\frac{\partial^3 T_o}{\partial r_o^3}\right|_{r_o=R_{in}}\\&-R_2\left.\frac{\partial^2 T_o}{\partial r_o^2}\right|_{r_o=R_{in}}-2\left.\frac{\sin\theta}{R_{in}^2}\frac{\partial T_1}{\partial \theta}\right|_{r_o=R_{in}}\end{aligned} \tag{5.167}$$

$$V_{z_2}\left(r_o = R_{out}\right) = 0 \tag{5.168}$$

and

$$V_{z_2}\left(r_o = R_{in}\right) = -2R_1 \left.\frac{dV_{z_1}}{dr_o}\right|_{r_o=R_{in}} - R_1^2 \left.\frac{d^2V_{z_o}}{dr_o^2}\right|_{r_o=R_{in}} - R_2 \left.\frac{dV_{z_o}}{dr_o}\right|_{r_o=R_{in}} \tag{5.169}$$

Now, let us find V_{z_o} ,V_{z_1} and V_{z_2} . Begin at order ε^o (*i.e.*, Equation (5.153)), where we have

$$\frac{1}{r_o}\frac{d}{dr_o}\left(r_o\frac{dV_{z_o}}{dr_o}\right) = 0$$

The solution of this equation must satisfy the boundary conditions given by Equations (5.156) and (5.157). Solving this equation, V_{z_o} is obtained as

$$V_{z_o} = \frac{V}{\ln\left(\frac{R_{in}}{R_{out}}\right)}\ln\left(\frac{r_o}{R_{out}}\right) = \alpha\ln\left(\frac{r_o}{R_{out}}\right) \tag{5.170}$$

where

$$\alpha = \frac{V}{\ln\left(\frac{R_{in}}{R_{out}}\right)} \tag{5.171}$$

Moreover,

$$\frac{dV_{z_o}}{dr_o} = \frac{\alpha}{r_o} \tag{5.172}$$

and

$$\frac{d^2V_{z_o}}{dr_o^2} = -\frac{\alpha}{r_o^2} \tag{5.173}$$

These derivatives will be useful later on.

At order ε^1, we must solve

$$\left[\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o\frac{\partial V_{z_1}}{\partial r_o}\right)+\frac{1}{r_o^2}\frac{\partial}{\partial \theta_o}\left(\frac{\partial V_{z_1}}{\partial \theta_o}\right)\right]=0$$

and V_{z_1} must satisfy the boundary conditions given by Equations (5.162) and (5.163).

Then, because $R_1 = \cos\theta_o$, V_{z_1} can be written as

$$V_{z_1} = \left(A_1 r_o + \frac{B_1}{r_o}\right)\cos\theta_o \tag{5.174}$$

where A_1 and B_1 must satisfy

$$A_1 R_{out} + \frac{B_1}{R_{out}} = 0 \tag{5.175}$$

and

$$A_1 R_{in} + \frac{B_1}{R_{in}} = -\frac{\alpha}{R_{in}} \tag{5.176}$$

and V_{z_1} is found to be

$$V_{z_1} = \alpha\frac{\cos\theta_o}{\left(R_{out}^2 - R_{in}^2\right)}\left(r_o - \frac{R_{out}^2}{r_o}\right) \tag{5.177}$$

whence

$$\frac{dV_{z_1}}{dr_o} = \alpha\frac{\cos\theta_o}{\left(R_{out}^2 - R_{in}^2\right)}\left(1 + \frac{R_{out}^2}{r_o^2}\right) \tag{5.178}$$

and

$$\frac{dV_{z_1}}{d\theta_o} = -\alpha\frac{\sin\theta_o}{\left(R_{out}^2 - R_{in}^2\right)}\left(r_o - \frac{R_{out}^2}{r_o}\right) \tag{5.179}$$

These derivatives will be useful later on.

To solve the V_{z_2} problem, first substitute Equations (5.172), (5.173) and (5.178) into Equation (5.169) and then expand the right-hand side of Equation (5.169) in terms of the functions[10] 1, $\cos\theta_o, \cos 2\theta_o$. Doing this Equation (5.169) reduces to

$$V_{z_2}(r_o = R_{in}) = \frac{-\alpha}{R_{in}^2\left(R_{out}^2 - R_{in}^2\right)}\left[2R_{in}^2 + \left(R_{out}^2 + R_{in}^2\right)\cos 2\theta_o\right] \tag{5.180}$$

Then, from Equations (5.168) and (5.180), the solution to the V_{z_2} problem can be written as

$$V_{z_2} = A_o + B_o \ln(r_o) + \left(A_2 r_o^2 + \frac{B_2}{r_o^2}\right)\cos 2\theta_o \tag{5.181}$$

where A_o and B_o must satisfy

$$A_o + B_o \ln(R_{out}) = 0 \tag{5.182}$$

and

$$A_o + B_o \ln(R_{in}) = \frac{-2\alpha}{\left(R_{out}^2 - R_{in}^2\right)} \tag{5.183}$$

and where A_2 and B_2 must satisfy

$$A_2 R_{out}^2 + \frac{B_2}{R_{out}^2} = 0 \tag{5.184}$$

and

$$A_2 R_{in}^2 + \frac{B_2}{R_{in}^2} = \frac{-\alpha\left(R_{out}^2 + R_{in}^2\right)}{R_{in}^2\left(R_{out}^2 - R_{in}^2\right)} \tag{5.185}$$

whence V_{z_2} is found to be

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[10] Substitute for $\cos^2\theta_o = \frac{1}{2} + \frac{1}{2}\cos 2\theta_o$ and for $\sin^2\theta_o = \frac{1}{2} - \frac{1}{2}\cos 2\theta_o$.

$$V_{z_2} = \frac{2\alpha}{\left(R_{out}^2 - R_{in}^2\right)\ln\left(\frac{R_{in}}{R_{out}}\right)}\ln\left(\frac{R_{out}}{r_o}\right) + \frac{\alpha}{\left(R_{out}^2 - R_{in}^2\right)^2}\left[r_o^2 - \frac{R_{out}^4}{r_o^2}\right]\cos 2\theta_o \qquad (5.186)$$

and by this, there obtains

$$\frac{dV_{z_2}}{dr_o} = \frac{-2\alpha}{r_o\left(R_{out}^2 - R_{in}^2\right)\ln\left(\frac{R_{in}}{R_{out}}\right)} + \frac{2\alpha}{\left(R_{out}^2 - R_{in}^2\right)^2}\left[r_o + \frac{R_{out}^4}{r_o^3}\right]\cos 2\theta_o \qquad (5.187)$$

This derivative will be needed later on.

Now, having obtained V_{z_o}, V_{z_1} and V_{z_2}, we can turn to Equations (5.152), (5.158) and (5.164) for T_o, T_1 and T_2.

Begin with the problem at order ε^o (*i.e.*, Equation (5.152)), namely

$$\frac{1}{r_o}\frac{d}{dr_o}\left(r_o\frac{dT_o}{dr_o}\right) + \frac{\mu}{k}\left(\frac{dV_{z_o}}{dr_o}\right)^2 = 0$$

which by using Equation (5.172) reduces to

$$\frac{d}{dr_o}\left(r_o\frac{dT_o}{dr_o}\right) + \frac{\mu\alpha^2}{kr_o} = 0 \qquad (5.188)$$

Solving this equation using the boundary conditions given by Equations (5.154) and (5.155) T_o is obtained as

$$T_o = \frac{\mu\alpha^2}{2k}\left[\left(\ln\left(R_{out}\right)\right)^2 - \left(\ln\left(r_o\right)\right)^2 + 2\ln\left(R_{in}\right)\ln\left(\frac{r_o}{R_{out}}\right)\right] \qquad (5.189)$$

From this we can get

$$\frac{dT_o}{dr_o} = \frac{\mu\alpha^2}{kr_o}\ln\left(\frac{R_{in}}{r_o}\right) \qquad (5.190)$$

$$\frac{d^2T_o}{dr_o^2} = \frac{\mu\alpha^2}{kr_o^2}\left[-1+\ln\left(\frac{r_o}{R_{in}}\right)\right] \tag{5.191}$$

and

$$\frac{d^3T_o}{dr_o^3} = \frac{\mu\alpha^2}{kr_o^3}\left[3+2\ln\left(\frac{R_{in}}{r_o}\right)\right] \tag{5.192}$$

which are going to be useful later on.

Of course Equation (5.189) is just the solution in the on-center case obtained earlier and given by Equation (5.148), namely

$$T = \frac{\delta^2}{2}\left[\left(\ln\left(R_{out}\right)\right)^2 - \left(\ln(r)\right)^2 + 2\ln\left(R_{in}\right)\ln\left(\frac{r}{R_{out}}\right)\right]$$

where δ^2 is just $\mu\alpha^2/k$.

Now, to find T_1 we must solve Equation (5.158) viz.,

$$\left[\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o\frac{\partial T_1}{\partial r_o}\right)+\frac{1}{r_o^2}\frac{\partial^2 T_1}{\partial \theta_o^2}\right]+2\frac{\mu}{k}\frac{\partial V_{z_o}}{\partial r_o}\frac{\partial V_{z_1}}{\partial r_o}=0$$

which by using Equations (5.172) and (5.178) we can rewrite it as

$$\left[\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o\frac{\partial T_1}{\partial r_o}\right)+\frac{1}{r_o^2}\frac{\partial^2 T_1}{\partial \theta_o^2}\right]+2\frac{\mu\alpha^2}{kr_o}\frac{\cos\theta_o}{\left(R_{out}^2-R_{in}^2\right)}\left(1+\frac{R_{out}^2}{r_o^2}\right)=0 \tag{5.193}$$

The solution of T_1 takes the form

$$T_1 = \hat{T}_1\cos\theta_o \tag{5.194}$$

where $\hat{T}_1$ must satisfy

$$\frac{d^2\hat{T}_1}{dr_o^2}+\frac{1}{r_o}\frac{d\hat{T}_1}{dr_o}-\frac{\hat{T}_1}{r_o^2}=f\left(r_o\right) \tag{5.195}$$

where

$$\hat{T}_1(r_o = R_{out}) = 0 \tag{5.196}$$

and

$$\frac{\partial \hat{T}_1}{\partial r_o}(r_o = R_{in}) = \frac{\mu\alpha^2}{kR_{in}^2} \tag{5.197}$$

and where

$$f\left(r_o\right) = -2\frac{\mu\alpha^2}{kr_o^3}\frac{\left(R_{out}^2 + r_o^2\right)}{\left(R_{out}^2 - R_{in}^2\right)} \tag{5.198}$$

The general solution of Equation (5.195) is the sum of a homogeneous solution, $\hat{T}_{1_{Homogeneous}}$, where $f\left(r_o\right)$ is set to zero, and a particular solution, $\hat{T}_{1_{Particular}}$. The homogeneous part is

$$\hat{T}_{1_{Homogeneous}} = Ar_o + \frac{B}{r_o} \tag{5.199}$$

and a particular solution will be

$$\hat{T}_{1_{Particular}} = \upsilon_1 r_o + \frac{\upsilon_2}{r_o} \tag{5.200}$$

where

$$\upsilon_1 = -\int_{R_{in}}^{r} \frac{f\left(r_o\right)}{w}\frac{1}{r_o}dr_o \tag{5.201}$$

and

$$\upsilon_2 = \int_{R_{in}}^{r} \frac{f\left(r_o\right)}{w} r_o dr_o \tag{5.202}$$

and where w is the Wronskian of r_o and $1/r_o$, which means

$$w = -\frac{2}{r_o} \tag{5.203}$$

Thus,

$$\upsilon_1 = \frac{1}{2}\int_{R_{in}}^{r} f\left(r_o\right) dr_o = -\frac{\mu\alpha^2}{k\left(R_{out}^2 - R_{in}^2\right)}\int_{R_{in}}^{r}\frac{\left(R_{out}^2 + r_o^2\right)}{r_o^3}dr_o \tag{5.204}$$

and

$$\upsilon_2 = -\frac{1}{2}\int_{R_{in}}^{r} f\left(r_o\right) r_o^2 dr_o = \frac{\mu\alpha^2}{k\left(R_{out}^2 - R_{in}^2\right)}\int_{R_{in}}^{r}\frac{\left(R_{out}^2 + r_o^2\right)}{r_o}dr_o \tag{5.205}$$

Hence

$$\hat{T}_1 = Ar_o + \frac{B}{r_o} - \frac{\mu\alpha^2}{k\left(R_{out}^2 - R_{in}^2\right)}\left[r_o\int_{R_{in}}^{r_o}\frac{\left(R_{out}^2 + r_o^2\right)}{r_o^3}dr_o - \frac{1}{r_o}\int_{R_{in}}^{r_o}\frac{\left(R_{out}^2 + r_o^2\right)}{r_o}dr_o\right] \tag{5.206}$$

where the constants A and B can be determined by using Equations (5.196) and (5.197). Doing this, $\hat{T}_1$ will turn out to be

$$\hat{T}_1 = \frac{\mu\alpha^2}{2k}\left(\frac{r_o^2 - R_{in}^2}{r_o R_{in}^2}\right) - \frac{\mu\alpha^2}{k\left(R_{out}^2 - R_{in}^2\right)}\left[r_o\int_{R_{in}}^{r_o}\frac{\left(R_{out}^2 + r_o^2\right)}{r_o^3}dr_o - \frac{1}{r_o}\int_{R_{in}}^{r_o}\frac{\left(R_{out}^2 + r_o^2\right)}{r_o}dr_o\right] \tag{5.207}$$

and whence $T_1 = \hat{T}_1 \cos\theta_o$ can be simplified and written as

$$T_1 = -\frac{\mu\alpha^2\left(r_o^2 - R_{in}^2\right)}{kr_o\left(R_{out}^2 - R_{in}^2\right)}\ln\left(\frac{r_o}{R_{in}}\right)\cos\theta_o \tag{5.208}$$

From this we can obtain

$$T_1\left(r_o = R_{in}\right) = 0 \tag{5.209}$$

$$\left.\frac{dT_1}{dr_o}\right|_{r_o = R_{in}} = \frac{\mu\alpha^2}{kR_{in}^2}\cos\theta_o \tag{5.210}$$

$$\left.\frac{d^2T_1}{dr_o^2}\right|_{r_o = R_{in}} = -\frac{\mu\alpha^2}{kR_{in}^3}\left(\frac{3R_{out}^2 + R_{in}^2}{R_{out}^2 - R_{in}^2}\right)\cos\theta_o \tag{5.211}$$

$$\left.\frac{d^3T_1}{dr_o^3}\right|_{r_o = R_{in}} = \frac{\mu\alpha^2}{kR_{in}^4}\left(\frac{11R_{out}^2 + R_{in}^2}{R_{out}^2 - R_{in}^2}\right)\cos\theta_o \tag{5.212}$$

$$\left.\frac{dT_1}{d\theta_o}\right|_{r_o=R_{in}}=0 \tag{5.213}$$

and

$$\left.\frac{d^2T_1}{d\theta_o^2}\right|_{r_o=R_{in}}=0 \tag{5.214}$$

These derivatives will be useful in determining T_2.

From Equation (5.209) we learn that to zero and first order T is just $T_o+\varepsilon R_1\left(dT_o/dr_o\right)$ where $R_1=\cos\theta_o$.

Now, to find T_2 we must solve Equation (5.164), which is

$$\left[\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o\frac{\partial T_2}{\partial r_o}\right)+\frac{1}{r_o^2}\frac{\partial^2 T_2}{\partial\theta_o^2}\right]+2\frac{\mu}{k}\left[\left(\frac{\partial V_{z_1}}{\partial r_o}\right)^2+\frac{\partial V_{z_o}}{\partial r_o}\frac{\partial V_{z_2}}{\partial r_o}+\frac{1}{r_o^2}\left(\frac{\partial V_{z_1}}{\partial\theta_o}\right)^2\right]=0$$

By using Equations (5.172), (5.178), (5.179) and (5.187), and by expanding $\cos^2\theta_o$ and $\sin^2\theta_o$ in terms of the functions 1, $\cos\theta_o,\cos 2\theta_o$ we can rewrite this equation as

$$\begin{aligned}\left[\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o\frac{\partial T_2}{\partial r_o}\right)+\frac{1}{r_o^2}\frac{\partial^2 T_2}{\partial\theta_o^2}\right]=&-\frac{2\mu\alpha^2\left(-2r_o^2\left(R_{out}^2-R_{in}^2\right)+\ln\left(\dfrac{R_{in}}{R_{out}}\right)\left(r_o^4-R_{in}^4\right)\right)}{kr_o^4\ln\left(\dfrac{R_{in}}{R_{out}}\right)\left(R_{out}^2-R_{in}^2\right)^2}\\&-\frac{4\mu\alpha^2\left(r_o^4+r_o^2R_{out}^2+R_{out}^4\right)}{kr_o^4\left(R_{out}^2-R_{in}^2\right)^2}\cos 2\theta_o\end{aligned} \tag{5.215}$$

where the boundary conditions to be satisfied are given by Equations (5.166) and (5.167) viz.,

$$T_2(r_o=R_{out})=0$$

and

$$\frac{\partial T_2}{\partial r_o}(r_o = R_{in}) = -2R_1 \left.\frac{\partial^2 T_1}{\partial r_o^2}\right|_{r_o=R_{in}} - R_1^2 \left.\frac{\partial^3 T_o}{\partial r_o^3}\right|_{r_o=R_{in}}$$
$$- R_2 \left.\frac{\partial^2 T_o}{\partial r_o^2}\right|_{r_o=R_{in}} - 2\frac{\sin\theta}{R_{in}^2}\left.\frac{\partial T_1}{\partial \theta}\right|_{r_o=R_{in}}$$

Equation (5.167) can be rewritten in a simpler form by using Equations (5.191), (5.192), (5.210) and (5.213) and expanding the right-hand side in terms of the functions $1, \cos\theta_o, \cos 2\theta_o$. Doing this Equation (5.167) reduces to

$$\frac{\partial T_2}{\partial r_o}(r_o = R_{in}) = \frac{\mu\alpha^2\left(R_{out}^2 + 3R_{in}^2\right)}{kR_{in}^3\left(R_{out}^2 - R_{in}^2\right)} + \frac{2\mu\alpha^2\left(R_{out}^2 + R_{in}^2\right)}{kR_{in}^3\left(R_{out}^2 - R_{in}^2\right)}\cos 2\theta_o \tag{5.216}$$

Now, the solution to Equation (5.215) can be written as

$$T_2 = \hat{T}_2 + \hat{\hat{T}}_2 \cos 2\theta_o \tag{5.217}$$

where $\hat{T}_2$ and $\hat{\hat{T}}_2$ satisfy

$$\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o \frac{\partial \hat{T}_2}{\partial r_o}\right) = -\frac{2\mu\alpha^2\left(-2r_o^2\left(R_{out}^2 - R_{in}^2\right) + \ln\left(\frac{R_{in}}{R_{out}}\right)\left(r_o^4 - R_{in}^4\right)\right)}{kr_o^4 \ln\left(\frac{R_{in}}{R_{out}}\right)\left(R_{out}^2 - R_{in}^2\right)^2} \tag{5.218}$$

$$\hat{T}_2(r_o = R_{out}) = 0 \tag{5.219}$$

and

$$\frac{\partial \hat{T}_2}{\partial r_o}(r_o = R_{in}) = \frac{\mu\alpha^2\left(R_{out}^2 + 3R_{in}^2\right)}{kR_{in}^3\left(R_{out}^2 - R_{in}^2\right)} \tag{5.220}$$

and

$$\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o \frac{\partial \hat{\hat{T}}_2}{\partial r_o}\right) - \frac{4}{r_o^2}\hat{\hat{T}}_2 = -\frac{4\mu\alpha^2\left(r_o^4 + r_o^2 R_{out}^2 + R_{out}^4\right)}{kr_o^4\left(R_{out}^2 - R_{in}^2\right)^2} \tag{5.221}$$

$$\hat{\hat{T}}_2(r_o = R_{out}) = 0 \tag{5.222}$$

and

$$\frac{\partial \hat{\hat{T}}_2}{\partial r_o}(r_o = R_{in}) = \frac{2\mu\alpha^2\left(R_{out}^2 + R_{in}^2\right)}{kR_{in}^3\left(R_{out}^2 - R_{in}^2\right)} \tag{5.223}$$

Solving these problems, T_2 turns out to be

$$\begin{aligned} T_2 = \mu\alpha^2 \Bigg[& 4r_o^2\left(R_{out}^2 + R_{in}^2\right)\ln\left(\frac{R_{out}}{r_o}\right)\ln\left(\frac{R_{in}}{R_{out}}\right) + 8r_o^2\left(R_{out}^2 - R_{in}^2\right)\ln\left(\frac{R_{out}}{r_o}\right)\ln\left(R_{in}\right) \\ & + 4r_o^2\left(R_{out}^2 - R_{in}^2\right)\left(\left(\ln\left(r_o\right)\right)^2 - \left(\ln\left(R_{out}\right)\right)^2\right) \\ & - \left(R_{out}^4 - 2R_{out}^2 r_o^2 + r_o^4\right)\ln\left(\frac{R_{in}}{R_{out}}\right)\Bigg] \Bigg/ \left[2kr_o^2\ln\left(\frac{R_{in}}{R_{out}}\right)\left(R_{out}^4 - 2R_{out}^2R_{in}^2 + R_{in}^4\right)\right] \\ & - \frac{\mu\alpha^2\left(2\ln\left(\frac{R_{in}}{r_o}\right)\left(R_{out}^4 - r_o^4\right) + R_{out}^4 - 2R_{out}^2r_o^2 + r_o^4\right)}{2kr_o^2\left(R_{out}^4 - 2R_{out}^2R_{in}^2 + R_{in}^4\right)}\cos 2\theta_o \end{aligned} \tag{5.224}$$

and from this we can obtain

$$\begin{aligned} \frac{dT_2}{dr_o} = \mu\alpha^2 \Bigg[& -\frac{2}{r_o}\left(R_{out}^2 + R_{in}^2\right)\ln\left(\frac{R_{in}}{R_{out}}\right) - \frac{4}{r_o}\left(R_{out}^2 - R_{in}^2\right)\ln\left(\frac{R_{in}}{r_o}\right) \\ & - \left(-\frac{R_{out}^4}{r_o^3} + r_o\right)\ln\left(\frac{R_{in}}{R_{out}}\right)\Bigg] \Bigg/ \left[k\ln\left(\frac{R_{in}}{R_{out}}\right)\left(R_{out}^4 - 2R_{out}^2R_{in}^2 + R_{in}^4\right)\right] \\ & + \frac{2\mu\alpha^2\left(R_{out}^4 - r_o^4\right)\left(1 + \ln\left(\frac{R_{in}}{r_o}\right)\right)}{kr_o^3\left(R_{out}^4 - 2R_{out}^2R_{in}^2 + R_{in}^4\right)}\cos 2\theta_o \end{aligned} \tag{5.225}$$

which will be used later.

The temperature profile is depicted in Figure 5-12 where we can see that the maximum temperature is located inside the thick region at the top of the shifted rod. This is expected because thick regions are bad heat conductors.

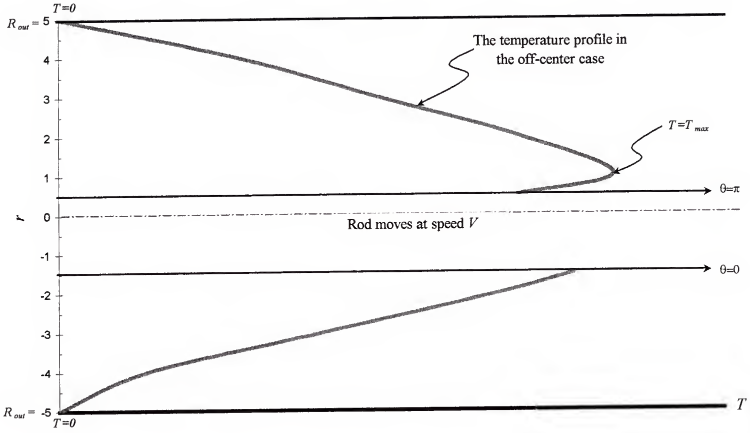

Figure 5-12. Temperature profile for off-center case.

Now, to study the effect of displacing the inner cylinder let us calculate the total volumetric-flow rate (*i.e.*, Q_f), and the total heat-transfer rate (*i.e.*, Q).

Let us begin with the volumetric-flow rate, Q_f, which can be determined via the formula

$$Q_f = \int_0^{2\pi} \int_{R(\theta,\varepsilon)}^{R_{out}} V_z(r,\theta,\varepsilon) r dr d\theta \tag{5.226}$$

where the integral is over the cross section of the displaced annulus. Carrying out the integration over the cross section of the reference annulus turns this formula into[11]

$$Q_f = \int_0^{2\pi} \int_{R_{in}}^{R_{out}} V_z r \frac{\partial(r,\theta)}{\partial(r_o,\theta_o)} dr_o d\theta_o \tag{5.227}$$

where

$$\frac{\partial(r,\theta)}{\partial(r_o,\theta_o)} = \frac{\partial r}{\partial r_o}\frac{\partial \theta}{\partial \theta_o} - \frac{\partial \theta}{\partial r_o}\frac{\partial r}{\partial \theta_o} \tag{5.228}$$

Then, by substituting this,

$$V_z = V_{z_o} + \varepsilon\left(V_{z_1} + r_1\frac{dV_{z_o}}{dr_o}\right) + \frac{\varepsilon^2}{2}\left(V_{z_2} + 2r_1\frac{dV_{z_1}}{dr_o} + r_1^2\frac{d^2V_{z_o}}{dr_o^2} + r_2\frac{dV_{z_o}}{dr_o}\right) + \ldots \tag{5.229}$$

and

$$r = r_o + \varepsilon r_1 + \frac{\varepsilon^2}{2} r_2 + \ldots \tag{5.230}$$

into Equation (5.227) we get

---

[11] For details on how to expand this integral see Appendix A.

$$
\begin{aligned}
Q_f = \int_0^{2\pi}\int_{R_{in}}^{R_{out}} \Bigg[& r_o V_{z_o} + \varepsilon\left(r_o V_{z_1} + \frac{\partial\left(r_o r_1 V_{z_o}\right)}{\partial r_o}\right) \\
& + \frac{\varepsilon^2}{2}\Bigg(2 r_o V_{z_2} + \frac{\partial\left(r_o r_1 V_{z_1}\right)}{\partial r_o} + 2\frac{\partial\left(r_o r_2 V_{z_o}\right)}{\partial r_o} + 2\frac{\partial\left(r_1^2 V_{z_o}\right)}{\partial r_o} \\
& + 2\frac{\partial}{\partial r_o}\left(r_o r_1^2 \frac{\partial V_{z_o}}{\partial r_o}\right)\Bigg)\Bigg] dr_o d\theta_o \equiv Q_{f_o} + \varepsilon Q_{f_1} + \frac{\varepsilon^2}{2} Q_{f_2}
\end{aligned}
\tag{5.231}
$$

where Q_{f_1} and Q_{f_2} are the corrections to Q_{f_o} due to the displacement of the inner cylinder.

Then, by substituting for V_{z_o}, V_{z_1} and V_{z_2} their values given in Equations (5.170), (5.177) and (5.186) the values of Q_{f_o}, Q_{f_1} and Q_{f_2} can be determined. Doing this Q_{f_o} is found to be

$$
\begin{aligned}
Q_{f_o} &= \alpha \int_0^{2\pi}\int_{R_{in}}^{R_{out}} \ln\left(\frac{r_o}{R_{out}}\right) r_o dr_o d\theta_o \\
&= -\frac{\alpha\pi}{2}\left(R_{out}^2 - R_{in}^2 + 2R_{in}^2 \ln\left(\frac{R_{in}}{R_{out}}\right)\right)
\end{aligned}
\tag{5.232}
$$

where

$$
\alpha = \frac{V}{\ln\left(\dfrac{R_{in}}{R_{out}}\right)}
$$

Then Q_{f_1} the first correction is found to be

$$
Q_{f_1} = \int_0^{2\pi}\int_{R_{in}}^{R_{out}} V_{z_1} r_o dr_o d\theta_o + \int_0^{2\pi} R_o R_1 V_{z_o}\Big|_{R_{in}}^{R_{out}} d\theta_o \tag{5.233}
$$

But this is zero as V_{z_1} and R_1 are multiples of $\cos\theta_o$. This is expected because we know intuitively that it should make no difference if the rod is shifted to the right (*i.e.*, $\varepsilon > 0$), or to the left (*i.e.*, $\varepsilon < 0$). Likeness, we expect all the odd Q_f's to be zero (*i.e.*, $Q_{f_3} = 0,\ Q_{f_5} = 0, \ldots$). Hence to learn the effect of a displacement we must go to second order of ε.

The value of Q_{f_2} is given by

$$
\begin{aligned}
Q_{f_2} = 2\int_0^{2\pi}\int_{R_{in}}^{R_{out}} V_{z_2} r_o dr_o d\theta_o + \int_0^{2\pi} R_o R_1 V_{z_1}\Big|_{R_{in}}^{R_{out}} d\theta_o + 2\int_0^{2\pi} R_o R_2 V_{z_o}\Big|_{R_{in}}^{R_{out}} d\theta_o \\
+ 2\int_0^{2\pi} R_1^2 V_{z_o}\Big|_{R_{in}}^{R_{out}} d\theta_o + 2\int_0^{2\pi} R_o R_1^2 \frac{\partial V_{z_o}}{\partial r_o}\Bigg|_{R_{in}}^{R_{out}} d\theta_o
\end{aligned}
\qquad (5.234)
$$

and because $V_{z_o}(r_o = R_{out}) = 0$, $V_{z_1}(r_o = R_{out}) = 0$ and $R_1(r_o = R_{out}) = 0$ the foregoing equation reduces to

$$
\begin{aligned}
Q_{f_2} = 2\int_0^{2\pi}\int_{R_{in}}^{R_{out}} V_{z_2} r_o dr_o d\theta_o + \int_0^{2\pi} R_o R_1 V_{z_1}\Big|_{R_{in}} d\theta_o + 2\int_0^{2\pi} R_o R_2 V_{z_o}\Big|_{R_{in}} d\theta_o \\
+ 2\int_0^{2\pi} R_1^2 V_{z_o}\Big|_{R_{in}} d\theta_o + 2\int_0^{2\pi} R_o R_1^2 \frac{\partial V_{z_o}}{\partial r_o}\Bigg|_{R_{in}} d\theta_o
\end{aligned}
\qquad (5.235)
$$

and on carrying out the integrations Q_{f_2} is equal to

$$
Q_{f_2} = \alpha\pi \left[\frac{\left(R_{out}^2 - R_{in}^2\right) - \left(R_{out}^2 - 3R_{in}^2\right)\ln\left(\dfrac{R_{in}}{R_{out}}\right)}{\left(R_{out}^2 - R_{in}^2\right)\ln\left(\dfrac{R_{in}}{R_{out}}\right)} \right] \qquad (5.236)
$$

which can be arranged in the following form

$$
Q_{f_2} = \alpha\pi \left[\frac{\left(R_{out}^2 - R_{in}^2\right)\left[1 - \ln\left(\dfrac{R_{in}}{R_{out}}\right)\right] - 2R_{in}^2 \ln\left(\dfrac{R_{in}}{R_{out}}\right)}{\left(R_{out}^2 - R_{in}^2\right)\ln\left(\dfrac{R_{in}}{R_{out}}\right)} \right] \qquad (5.237)
$$

From this equation it becomes obvious that Q_{f_2} is always positive. Hence we observe that the total volumetric-flow rate for the displaced rod ($Q_f = Q_{f_o} + \varepsilon^2 Q_{f_2}/2$), is greater than it is for on-centered one by an amount equal to $\varepsilon^2 Q_{f_2}/2$. From this, we can conclude that when the inner cylinder is perturbed as shown in Figure 5-11B the volumetric-flow rate can change only at order ε^2 and that the total volumetric-flow rate increases. Therefore, the displaced rod causes more flow than the on-centered rod at the same velocity. To explain this, we are going to calculate the total shear stress and the total power applied to the fluid by the rod for the two cases.

The shear stress for the on-centered case is given by

$$\tau_{rz} = \mu \frac{dV_z}{dr} = \mu \frac{dV_{z_o}}{dr_o} \tag{5.238}$$

and the shear stress for the off-centered case is given by

$$\begin{aligned} \tau_{rz} + \tau_{\theta z} = \mu\left[\frac{\partial V_z}{\partial r} + \frac{1}{r}\frac{\partial V_z}{\partial \theta}\right] = \mu\Bigg[&\frac{\partial V_{z_o}}{\partial r_o} + \varepsilon\left(\frac{\partial V_{z_1}}{\partial r_o} + R_1 \frac{\partial^2 V_{z_o}}{\partial r_o^2}\right) \\ &+ \frac{\varepsilon^2}{2}\left(\frac{\partial V_{z_2}}{\partial r_o} + 2R_1 \frac{\partial^2 V_{z_1}}{\partial r_o^2} + R_1^2 \frac{\partial^3 V_{z_o}}{\partial r_o^3} + R_2 \frac{\partial^2 V_{z_o}}{\partial r_o^2}\right) \\ &+ \frac{1}{r_o}\left(\frac{\partial V_{z_o}}{\partial \theta_o} + \varepsilon\left(\frac{\partial V_{z_1}}{\partial \theta_o} + R_1 \frac{\partial^2 V_{z_o}}{\partial \theta_o^2}\right)\right. \\ &\left.\left.+ \frac{\varepsilon^2}{2}\left(\frac{\partial V_{z_2}}{\partial \theta_o} + 2R_1 \frac{\partial^2 V_{z_1}}{\partial \theta_o^2} + R_1^2 \frac{\partial^3 V_{z_o}}{\partial \theta_o^3} + R_2 \frac{\partial^2 V_{z_o}}{\partial \theta_o^2}\right)\right)\right] \end{aligned} \tag{5.239}$$

To calculate the total power applied to the fluid by the rod let us consider Equation (5.139) viz.,

$$\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{\partial V_z}{\partial \theta}\right)\right] = 0$$

In the on-centered case we know that V_z is independent of θ, whence this equation can be written

$$\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_z}{\partial r}\right)=0 \tag{5.240}$$

Now, let us multiply by V_z and integrate over the cross section of the flow to get

$$\int_0^{2\pi}\int_{R_{in}}^{R_{out}}\mu V_z\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_z}{\partial r}\right)rdrd\theta=0 \tag{5.241}$$

On integrating Equation (5.241) by parts we find

$$\int_0^{2\pi}\mu V_z\frac{\partial V_z}{\partial r}\bigg|_{R_{in}}^{R_{out}}d\theta-\int_0^{2\pi}\int_{R_{in}}^{R_{out}}\mu\left(\frac{\partial V_z}{\partial r}\right)^2 rdrd\theta=0 \tag{5.242}$$

and as $V_z\left(r=R_{out}\right)=0$ this reduces to

$$-\int_0^{2\pi}\mu V_z\frac{\partial V_z}{\partial r}\bigg|_{R_{in}}d\theta=\int_0^{2\pi}\int_{R_{in}}^{R_{out}}\mu\left(\frac{\partial V_z}{\partial r}\right)^2 rdrd\theta=\int_0^{2\pi}\int_{R_{in}}^{R_{out}}\mu\left(\frac{\partial V_{z_o}}{\partial r_o}\right)^2 r_o dr_o d\theta_o \tag{5.243}$$

The left-hand side is the power put into the fluid by the rod while the right-hand side is rate at which the fluid dissipates mechanical energy. Now, let us consider the case where the rod is displaced. In this case, let us multiply Equation (5.139) by V_z and integrate over the cross section of the annulus. From this we get

$$\int_0^{2\pi}\int_{R_{in}}^{R_{out}}\left(\mu\frac{V_z}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_z}{\partial r}\right)+\mu\frac{V_z}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\partial V_z}{\partial\theta}\right)\right)rdrd\theta=0 \tag{5.244}$$

and integrating by parts we find

$$\begin{aligned}&\int_0^{2\pi}\mu V_z\frac{\partial V_z}{\partial r}\bigg|_{R_{in}}^{R_{out}}d\theta-\int_0^{2\pi}\int_{R_{in}}^{R_{out}}\left(\mu\left(\frac{\partial V_z}{\partial r}\right)^2\right)rdrd\theta\\&\qquad+\int_{R_{in}}^{R_{out}}\mu\frac{V_z}{r^2}\frac{\partial V_z}{\partial\theta}\bigg|_0^{2\pi}dr-\int_0^{2\pi}\int_{R_{in}}^{R_{out}}\left(\frac{\mu}{r^2}\left(\frac{\partial V_z}{\partial\theta}\right)^2\right)rdrd\theta=0\end{aligned} \tag{5.245}$$

and as the third integral vanishes we get

$$-\int_{0}^{2\pi} \mu V_z \frac{\partial V_z}{\partial r}\bigg|_{R_{in}} d\theta = \int_{0}^{2\pi}\int_{R_{in}}^{R_{out}} \mu\left(\left(\frac{\partial V_z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial V_z}{\partial \theta}\right)^2\right) r dr d\theta \tag{5.246}$$

or

$$\begin{aligned} -\int_{0}^{2\pi} \mu V_z \frac{\partial V_z}{\partial r}\bigg|_{R_{in}} d\theta = \int_{0}^{2\pi}\int_{R_{in}}^{R_{out}} \mu \Bigg[& \Bigg(\frac{\partial V_{z_o}}{\partial r_o} + \varepsilon\left(\frac{\partial V_{z_1}}{\partial r_o} + R_1 \frac{\partial^2 V_{z_o}}{\partial r_o^2}\right) \\ & + \frac{\varepsilon^2}{2}\left(\frac{\partial V_{z_2}}{\partial r_o} + 2R_1 \frac{\partial^2 V_{z_1}}{\partial r_o^2} + R_1^2 \frac{\partial^3 V_{z_o}}{\partial r_o^3} + R_2 \frac{\partial^2 V_{z_o}}{\partial r_o^2}\right)\Bigg)^2 \\ & + \frac{1}{r_o^2}\Bigg(\frac{\partial V_{z_o}}{\partial \theta_o} + \varepsilon\left(\frac{\partial V_{z_1}}{\partial \theta_o} + R_1 \frac{\partial^2 V_{z_o}}{\partial \theta_o^2}\right) \\ & + \frac{\varepsilon^2}{2}\left(\frac{\partial V_{z_2}}{\partial \theta_o} + 2R_1 \frac{\partial^2 V_{z_1}}{\partial \theta_o^2} + R_1^2 \frac{\partial^3 V_{z_o}}{\partial \theta_o^3} + R_2 \frac{\partial^2 V_{z_o}}{\partial \theta_o^2}\right)\Bigg)^2\Bigg] r_o dr_o d\theta_o \end{aligned} \tag{5.247}$$

Once more the left-hand side is the power put into the fluid by the displaced rod while the right-hand side is rate at which the fluid dissipates mechanical energy. The results obtained from this calculations for given values of R_{out}, R_{in}, ε and V are illustrated in Table 5-2.

Table 5-2. Total shear stress and the total power input for on-centered and off-centered cases at $R_{out} = 5$, $R_{in} = 1$, $\varepsilon = 0.5$ and V=10.

	On-centered	Off-centered
Total shear stress $\left(\frac{\text{M}}{\text{Lt}^2}\right)$	12.43 μ	21.62 μ
Total power input $\left(\frac{\text{M}}{\text{t}^3}\right)$	390.4 μ	406.0 μ

Now from Table 5-2 we can notice that when we shift the rod the total shear stress will increase and hence it becomes harder to drag the rod along the pipe. Therefore, more energy needs to be applied to drag the rod at the same velocity. This energy will dissipate in the region causing the fluid to flow faster and, hence, the temperature to increase. This increase in temperature will reduce the thermal stability of the system.

Now let us turn to the total heat-transfer rate (*i.e.*, Q). This can be determined in two ways. The first way is to calculate the total heat dissipated to the surrounding via

$$Q = \int dA\vec{n} \cdot \vec{q} \quad , \quad \text{integral over the outer wall} \qquad (5.248)$$

where $\vec{q}$ is the heat flux and where $\vec{n}$ is the unit outward normal to the annular region. And the second way is to calculate the total heat produced due to viscous heating, which is given by

$$Q = \int_{0}^{2\pi} \int_{R(\theta,\varepsilon)}^{R_{ut}} \frac{1}{\mu} \left[{\tau_{rz}}^2 + {\tau_{\theta z}}^2 \right] d\theta \qquad (5.249)$$

These two ways must give the same results. However, the first method is not new to us as it was discussed and used earlier in Chapter 2 and because of that we are going to carry out this calculation by using the second method.

To begin let us write Equation (5.249) in terms of the velocity. Doing this we get

$$Q = \mu \int_{0}^{2\pi} \int_{R(\theta,\varepsilon)}^{R_{ut}} \left[\left(\frac{\partial V_z}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial V_z}{\partial \theta} \right)^2 \right] r dr d\theta \qquad (5.250)$$

where the integral is over the cross section of the displaced annulus. Carrying out the integral over the cross section of the reference annulus turns this formula into[12]

$$Q = \mu \int_{0}^{2\pi} \int_{R_{in}}^{R_{out}} r \left[\left(\frac{\partial V_z}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial V_z}{\partial \theta} \right)^2 \right] \frac{\partial (r,\theta)}{\partial (r_o,\theta_o)} dr_o d\theta_o \qquad (5.251)$$

whereupon substituting Equations (5.229) and (5.230), namely

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[12] Appendix A gives full details on how to expand this integral.

$$V_z = V_{z_o} + \varepsilon\left(V_{z_1} + r_1\frac{dV_{z_o}}{dr_o}\right) + \frac{\varepsilon^2}{2}\left(V_{z_2} + 2r_1\frac{dV_{z_1}}{dr_o} + r_1^{\,2}\frac{d^2V_{z_o}}{dr_o^2} + r_2\frac{dV_{z_o}}{dr_o}\right) + \dots$$

and

$$r = r_o + \varepsilon r_1 + \frac{\varepsilon^2}{2} r_2 + \dots$$

into Equation (5.251) we obtain

$$\begin{aligned} Q = \mu \int_0^{2\pi}\int_{R_{in}}^{R_{out}} \Bigg[& r_o\left(\frac{\partial V_{z_o}}{\partial r_o}\right)^2 + \varepsilon\left(2r_o\frac{\partial V_{z_o}}{\partial r_o}\frac{\partial V_{z_1}}{\partial r_o} + \frac{\partial}{\partial r_o}\left(r_o r_1\left(\frac{\partial V_{z_o}}{\partial r_o}\right)^2\right)\right) \\ & + \varepsilon^2\left(\frac{\partial}{\partial r_o}\left(\frac{1}{2}r_o r_2\left(\frac{\partial V_{z_o}}{\partial r_o}\right)^2\right) + \frac{\partial}{\partial r_o}\left(r_o r_1^{\,2}\frac{\partial V_{z_o}}{\partial r_o}\frac{\partial^2 V_{z_o}}{\partial r_o^2}\right)\right. \\ & + \frac{\partial}{\partial r_o}\left(\frac{1}{2}r_1^{\,2}\left(\frac{\partial V_{z_o}}{\partial r_o}\right)^2\right) + \frac{\partial}{\partial r_o}\left(2r_o r_1\frac{\partial V_{z_o}}{\partial r_o}\frac{\partial V_{z_1}}{\partial r_o}\right) \\ & \left. + \frac{1}{r_o}\left(\frac{\partial V_{z_1}}{\partial \theta_o}\right)^2 + r_o\left(\frac{\partial V_{z_1}}{\partial r_o}\right)^2 + r_o\frac{\partial V_{z_o}}{\partial r_o}\frac{\partial V_{z_2}}{\partial r_o}\right)\Bigg] dr_o\, d\theta_o \end{aligned} \tag{5.252}$$

or Q can be written as

$$Q = Q_o + \varepsilon Q_1 + \varepsilon^2 Q_2 + \dots \tag{5.253}$$

where Q_o is the total heat-transfer rate for the symmetric case and Q_1 and Q_2 are the corrections to Q_o due to the displacement of the rod.

Now, let us substitute for V_{z_o}, V_{z_1} and V_{z_2} their values obtained earlier into Equation (5.252). Doing this Q_o is found to be

$$Q_o = \mu\alpha^2 \int_0^{2\pi}\int_{R_{in}}^{R_{out}} \frac{1}{r_o} dr_o\, d\theta_o = 2\pi\mu\alpha^2 \ln\left(\frac{R_{out}}{R_{in}}\right) > 0 \tag{5.254}$$

Then Q_1 the first correction is found to be

$$Q_1 = \mu \int_0^{2\pi} \int_{R_{in}}^{R_{out}} 2r_o \frac{\partial V_{z_o}}{\partial r_o} \frac{\partial V_{z_1}}{\partial r_o} dr_o d\theta_o + \mu \int_0^{2\pi} R_o R_1 \left(\frac{\partial V_{z_o}}{\partial r_o} \right)^2 \Bigg|_{R_{in}}^{R_{out}} d\theta_o \tag{5.255}$$

But this is zero because $\left(\partial V_{z_1} / \partial r_o \right)$ and R_1 are multiples of $\cos \theta_o$. This is an expected result and it happened due to the symmetry of the annulus. Likeness, we expect all the odd Q's to be zero. Hence to see the effect of a displacement we ought to go to the second order of ε.

The value of Q_2 is given by

$$\begin{aligned} Q_2 = {} & \mu \int_0^{2\pi} \frac{1}{2} R_o R_2 \left(\frac{\partial V_{z_o}}{\partial r_o} \right)^2 \Bigg|_{R_{in}}^{R_{out}} d\theta_o + \mu \int_0^{2\pi} R_o R_1^2 \frac{\partial V_{z_o}}{\partial r_o} \frac{\partial^2 V_{z_o}}{\partial r_o^2} \Bigg|_{R_{in}}^{R_{out}} d\theta_o \\ & + \mu \int_0^{2\pi} \frac{1}{2} R_1^2 \left(\frac{\partial V_{z_o}}{\partial r_o} \right)^2 \Bigg|_{R_{in}}^{R_{out}} d\theta_o + \mu \int_0^{2\pi} 2 R_o R_1 \frac{\partial V_{z_o}}{\partial r_o} \frac{\partial V_{z_1}}{\partial r_o} \Bigg|_{R_{in}}^{R_{out}} d\theta_o \\ & + \mu \int_0^{2\pi} \int_{R_{in}}^{R_{out}} \left(\frac{1}{r_o} \left(\frac{\partial V_{z_1}}{\partial \theta_o} \right)^2 + r_o \left(\frac{\partial V_{z_1}}{\partial r_o} \right)^2 + r_o \frac{\partial V_{z_o}}{\partial r_o} \frac{\partial V_{z_2}}{\partial r_o} \right) dr_o d\theta_o \end{aligned} \tag{5.256}$$

and on carrying out the integrations Q_2 is equal to

$$Q_2 = \frac{\mu \alpha^2 \pi \left(2R_{out}^3 + 3R_{out}^2 R_{in} + R_{in}^3 \right)}{R_{out}^3 \left(R_{out}^2 - R_{in}^2 \right)} \tag{5.257}$$

Notice that the value of Q_2 given by Equation (5.257) is always positive. Hence, we observe that the total heat-transfer rate for the displaced rod ($Q = Q_o + \varepsilon^2 Q_2 / 2$), is greater than it is for an on-centered one by an amount equal to $\varepsilon^2 Q_2 / 2$. This happens due to the effect of the azimuthal gradients. From this we can conclude that, when the inner cylinder is perturbed as shown in Figure 5-11B it is clear that the heat-transfer rate can change only at order ε^2 and that the total heat-transfer rate increases. Therefore, the displaced rod can tolerate more heat generation because it appears to be a better heat conductor than the on-centered one.

Now, from determining the total volumetric-flow rate (*i.e.*, Q_f), and the total heat-transfer rate (*i.e.*, Q), for this problem we learned that the two rates will increase as the rod is shifted slightly from the center of the annulus. And also we learned that the increase in these two rates have two opposite effects on the thermal stability of this problem (*i.e.*, the increase in total volumetric-flow rate will decrease the thermal stability of the problem while the increase in the total heat-transfer rate will make the problem more stable). But we know that this problem is always thermally stable then, it would appear that either the two effects just offset leaving the thermal stability of the problem unchanged or the stabilizing effect (*i.e.*, the increase in the total heat-transfer rate), will be the predominant effect. Notice that even in this case where the viscosity is constant the calculations are not ordinarily simple.

Now, let us summarize what we have done and learned in this chapter. The aim of this chapter was to illustrate how the results presented in earlier chapters tie into some concrete physical problems. To achieve this, we considered viscous heating of liquids in simple flows and we started by showing that viscous fluid flow problems are in fact thermal ignition problems in which the heat source depends linearly on temperature. Then, some auxiliary problems were studied and solved to get some idea of the effect of unequal temperatures at the boundaries and of the effect of geometry change in frictional heating. From these auxiliary problems we learned that, the thermal stability of viscous heating does not depend on whether the two boundaries are at different temperatures or not. Even though the details of the flow and the temperature will be affected by the temperature difference at the boundaries. In addition to this we learned that the thermal stability of viscous heating problems is strongly dependent on their geometry. After that we turned to the main problem of this chapter (*i.e.*, thermal stability of frictional heating in axial flow between a rod and a pipe due to viscous heating). On studying this problem we found that, this problem can be solved only when it is perfectly symmetric but as the inner rod is shifted slightly from the center of the annulus the mathematics of the problem become very complicated. However, to learn more about thermal stability of this geometry we carried out the calculation but with approximation. This approximation is to hold the viscosity fixed. By solving this modified problem we found that, this problem is always thermally stable and when the inner cylinder is displaced slightly from the center of the annulus we found that both the total volumetric-flow rate and the total heat-transfer rate increase. The increase in these two rates have two opposite effects on the thermal stability of this problem (*i.e.*, the increase in total volumetric-flow rate reduces the thermal stability of the problem while the increase in the total heat-transfer rate enhances its thermal stability). It would appear that either the two effects just offset leaving the thermal stability of the problem unchanged or the stabilizing effect will be the predominant effect. Also we found that the location of the maximum temperature strongly depends on the shape of the geometry. In addition to that we learnt that even for this simple problem the calculations that determine the total volumetric-flow rate and the total heat-transfer rate are not ordinarily simple.

In the next chapter (Chapter 6), another practical problem is considered. This problem is to study the thermal stability in porous media with natural convection.

CHAPTER 6
THERMAL STABILITY IN POROUS MEDIA

In this chapter we retain our interest in heat generation and in the sizes and the shapes of objects, which can dissipate their heat while the temperature rise in the object remains finite.

Two kinds of problems will be of interest. In the first we ask whether an adverse temperature gradient on the way to thermal instability, due to conduction alone, can at some point induce convection that will postpone thermal instability. In the second we come back to our interest in the effect of perturbing the geometry of the object.

To study the first problem let us consider a porous material confined in an annular region between a rod and a pipe and let us assume that the porous material is producing heat and further assume that the rod and the pipe walls are insulated while the top and the bottom walls of the annular region are impermeable and held at a fixed temperature. The annular region has a finite height denoted H and the radii of the rod and the pipe are denoted R_{in} and R_{out}, respectively. The heat source will be taken to depend linearly on the temperature of the heat-generating material. This problem is depicted in Figure 6-1. The thermal conductivity is denoted k_c, the fluid filling the pores has a viscosity denoted μ and a density denoted ρ. The density depends on temperature and we write

$$\rho = \rho_{ref}\left[1-\alpha\left(T-T_{ref}\right)\right], \quad \alpha > 0 \tag{6.1}$$

where α denotes the coefficient of thermal expansion, T is the temperature and T_{ref} is

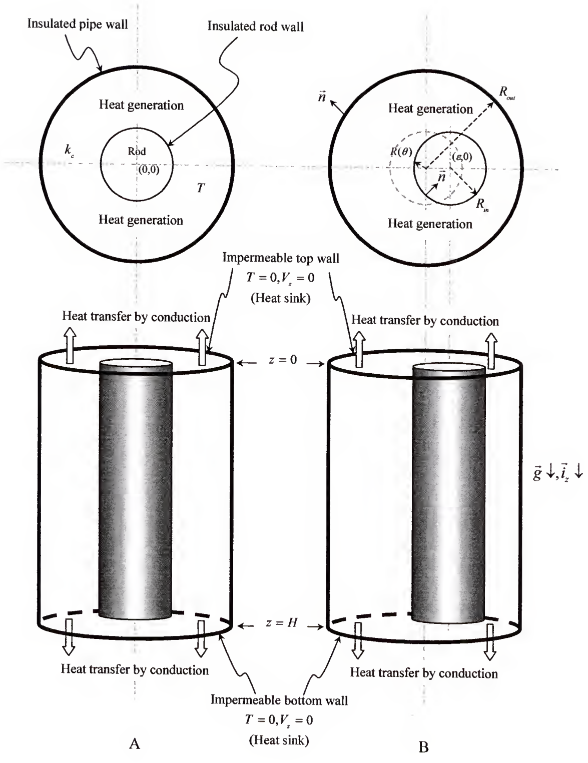


Figure 6-1. Annulus with heat-producing porous material. A) On-center. B) Off-center.

the temperature at which $\rho = \rho_{ref}$. This definition for the density of the fluid expresses the main idea that hot fluid is less dense than cold fluid. And because the porous material is generating heat the formation of a hot spot is expected. The hot spot will cause density differences inside the heated region by which there will be a tendency on part of the fluid to redistribute itself (*i.e.*, to flow). Now, without flow, there is a critical value of the strength of the heat source. But if in the course of heat conduction the temperature variation induces flow, and perhaps this flow will allow a greater strength of heat generation. To find out whether or not the fluid filling the pores will begin to flow as the region heats up requires a calculation. This calculation will be presented in the coming section.

Now, to study the second problem let us displace the rod a distance ε off-center. This shift in the rod, as depicted in Figure 6-1B, will create both a thick and a thin heat-producing region. The thick region is dangerous. To find out whether off-center or on-center rods are more dangerous requires a calculation. This calculation is presented in a subsequent section.

Thermal Stability of a Heat-Generating Porous Material between a Pipe and a Rod. In the on-center case the temperature rise of the porous material must satisfy

$$\rho C_v \frac{\partial T}{\partial t} + \rho C_v \vec{V} \cdot \nabla T = k_c \nabla^2 T + Q(T) \tag{6.2}$$

where ρ and C_v denote the density and the heat capacity of the heat-producing material, respectively, T is the temperature, $\vec{V}$ is the velocity of the fluid and $Q(T)$ denotes the rate at which heat is produced in the material per unit volume. The velocity of the fluid is given by Darcy's law, viz.,

$$\vec{V} = \frac{k_p}{\mu}\left(-\nabla P + \rho\vec{g}\right) \tag{6.3}$$

where k_p denotes the permeability of the porous medium, μ denotes the viscosity of the fluid, ∇P is the pressure gradient and $\rho\vec{g}$ is the gravitational force per unit volume. The equation of continuity is given by

$$\nabla \cdot \vec{V} = 0 \tag{6.4}$$

The boundary conditions to be satisfied are

$$T\left(z=0\right)=0\ , \quad \text{at the top wall} \tag{6.5}$$

$$T\left(z=H\right)=0\ , \quad \text{at the bottom wall} \tag{6.6}$$

$$V_z\left(z=0\right)=0\ , \quad \text{at the top wall} \tag{6.7}$$

$$V_z\left(z=H\right)=0\ , \quad \text{at the bottom wall} \tag{6.8}$$

$$\vec{n}.\nabla T\left(r=R_{in}\right)=\frac{dT}{dr}\left(r=R_{in}\right)=0\ , \quad \text{at the rod wall} \tag{6.9}$$

$$\vec{n}.\nabla T\left(r=R_{out}\right)=\frac{dT}{dr}\left(r=R_{out}\right)=0\ , \quad \text{at the pipe wall} \tag{6.10}$$

$$\vec{n}\cdot\vec{V}\left(r=R_{in}\right)=V_r\left(r=R_{in}\right)=0\ , \quad \text{at the rod wall} \tag{6.11}$$

and

$$\vec{n}\cdot\vec{V}\left(r=R_{out}\right)=V_r\left(r=R_{out}\right)=0\ , \quad \text{at the pipe wall} \tag{6.12}$$

where $\vec{n}$ is the unit outward normal to the annular region.

Now, the density, which appears in the domain equations, varies with the temperature of the fluid according to Equation (6.1). But we are really interested in studying the effect of density as a source of flow, so we are going to allow the density to be a variable only where it can cause flow. Consequently, to get a simple model we are going to hold the density fixed everywhere in this model except in the gravity term.

Let $Q(T) = Q_o + Q_1(T - T_w)$ and set $\dot{T} = T - T_w$ then $\dot{T}$ must satisfy

$$\rho C_v \frac{\partial \dot{T}}{\partial t} + \rho C_v \vec{V} \cdot \nabla \dot{T} = k_c \nabla^2 \dot{T} + Q_o \left(1 + S\dot{T}\right) \tag{6.13}$$

where

$$S = \frac{Q_1}{Q_o}$$

and using

$$\rho = \rho_{ref}\left[1 - \alpha\left(T - T_{ref}\right)\right]$$

$\vec{V}$ must satisfy

$$\vec{V} = \frac{k_p}{\mu}\left(-\nabla \mathbb{P} - \rho_{ref}\alpha \vec{g}\left(T - T_{ref}\right)\right) \tag{6.14}$$

where

$$-\nabla \mathbb{P} = -\nabla P + \rho_{ref}\vec{g} \tag{6.15}$$

Equations (6.4), (6.13) and (6.14) can be written in scaled form by defining $\dot{T} = \overline{T}\ T^*$, $\vec{V} = \overline{V}\ \vec{V}^*$, $\nabla = \overline{\nabla}\ \nabla^*$, $z = \overline{z}\ z^*$, $r = \overline{r}\ r^*$, $\mathbb{P} = \overline{\mathbb{P}}\ \mathbb{P}^*$ and $t = \overline{t}\ t^*$, where $\overline{T} = 1/S$, $\overline{V} = k_c/\rho C_v R_{in}$, $\overline{\nabla} = 1/R_{in}$, $\overline{z} = R_{in}$, $\overline{r} = R_{in}$, $H^* = H/R_{in}$, $R^*_{out} = R_{out}/R_{in}$, $\overline{\mathbb{P}} = \mu k_c/k_p \rho C_v$ and $\overline{t} = \rho C_v R_{in}^2/k_c$. Then, on dropping the *'s, the equations to be solved are

$$\frac{\partial T}{\partial t} + \vec{V} \cdot \nabla T = \nabla^2 T + \delta^2\left(1 + T\right) \tag{6.16}$$

$$\vec{V} = -\nabla \mathbb{P} - RaT\vec{i}_z \tag{6.17}$$

and

$$\nabla \cdot \vec{V} = 0 \tag{6.18}$$

where

$$\delta^2 = \frac{Q_o R_{in}^2 S}{k_c},\ Ra = \frac{\alpha k_p R_{in} g}{k_d \nu S},\ k_d = \frac{k_c}{\rho C_v} \text{ and } \nu = \frac{\mu}{\rho_{ref}}$$

and where Ra is called the *Rayleigh number*. The boundary conditions to be satisfied are

$$T(z=0) = 0\ , \quad \text{at the top wall} \tag{6.19}$$

$$T(z=H) = 0\ , \quad \text{at the bottom wall} \tag{6.20}$$

$$V_z(z=0) = 0\ , \quad \text{at the top wall} \tag{6.21}$$

$$V_z(z=H) = 0\ , \quad \text{at the bottom wall} \tag{6.22}$$

$$\vec{n}.\nabla T(r=1) = \frac{dT}{dr}(r=1) = 0\ , \quad \text{at the rod wall} \tag{6.23}$$

$$\vec{n}.\nabla T(r=R_{out}) = \frac{dT}{dr}(r=R_{out}) = 0\ , \quad \text{at the pipe wall} \tag{6.24}$$

$$\vec{n} \cdot \vec{V}(r=1) = V_r(r=1) = 0\ , \quad \text{at the rod wall} \tag{6.25}$$

and

$$\vec{n} \cdot \vec{V}(r=R_{out}) = V_r(r=R_{out}) = 0\ , \quad \text{at the pipe wall} \tag{6.26}$$

First notice that Ra is not a control variable, it depends on the properties of the porous medium, the fluid and the geometry. The value of Ra will increase as the heat source becomes stronger (strength the density gradient), the permeability increases (fluid flows easily through big gaps), the thermal expansion increases (strength the density gradient), the thermal diffusivity decreases (hard to eliminate the horizontal temperature perturbation), the gravity increases (strength the density gradient), the kinematic viscosity decreases (less resistance to flow) and as the sensitivity increases (strength the density gradient). Second there appear to be two input variables, δ^2 and Ra. But it is possible to get a simple no flow solution to the problem for all values of Ra and for each value of δ^2 up to $\delta^2_{critical}$ at no flow.

Our first job then will be to find the critical value of Ra, at which the steady conduction solution goes instable, as a function of δ^2.

In the case $\vec{V}_o = 0$, the steady conduction solution satisfies

$$\frac{d^2 T_o}{dz^2} + \delta^2 \left(1 + T_o\right) = 0 \tag{6.27}$$

and

$$-\frac{d\mathbb{P}_o}{dz} = Ra T_o \tag{6.28}$$

where

$$T_o\left(z = 0\right) = 0 \ , \quad \text{at the top wall} \tag{6.29}$$

and

$$T_o\left(z = H\right) = 0 \ , \quad \text{at the bottom wall} \tag{6.30}$$

whence T_o is given by

$$T_o = A\cos\delta z + B\sin\delta z - 1 \tag{6.31}$$

where

$$A = 1 \tag{6.32}$$

and

$$A\cos\delta H + B\sin\delta H = 1 \tag{6.33}$$

whereupon

$$T_o = \frac{\cos\delta z \sin\delta H + \left(1-\cos\delta H\right)\sin\delta z}{\sin\delta H} - 1 \tag{6.34}$$

and by this, there obtains

$$\frac{dT_o}{dz} = \delta\frac{-\sin\delta z \sin\delta H + \left(1-\cos\delta H\right)\cos\delta z}{\sin\delta H} \tag{6.35}$$

which will be useful later on. The base variation of $\mathbb{P}$ can be found but it will not be needed. The base solution holds for any value of Ra and it is also tells us the critical value of δ^2 at no flow, which means

$$\delta_{critical} = \frac{\pi}{H}\,, \quad \text{at no flow} \tag{6.36}$$

Now, the base temperature induces a density variation, which is high at the top and the bottom and low in between. The adverse gradient from the top to the middle may induce a flow and to find out we determine the critical value of Ra at which the no flow solution loses its stability to velocity perturbation. To find $Ra_{critical}$ we are going to perturb the domain equations about the base solution in the usual way. Hence we are going to write

$$T = T_o + \epsilon T_1 \tag{6.37}$$

$$\vec{V} = \vec{V}_o + \epsilon \vec{V}_1 \tag{6.38}$$

and

$$\mathbb{P} = \mathbb{P}_o + \epsilon \mathbb{P}_1 \tag{6.39}$$

whence the perturbation problem becomes

$$\frac{\partial T_1}{\partial t} + \vec{V}_1 \cdot \nabla T_o = \nabla^2 T_1 + \delta^2 T_1 \tag{6.40}$$

$$\vec{V}_1 = -\nabla \mathbb{P}_1 - Ra T_1 \vec{i}_z \tag{6.41}$$

and

$$\nabla \cdot \vec{V}_1 = 0 \tag{6.42}$$

where the boundary conditions to be satisfied are

$$T_1(z=0)=0 \tag{6.43}$$

$$T_1(z=H)=0 \tag{6.44}$$

$$V_{z_1}(z=0)=0 \tag{6.45}$$

$$V_{z_1}(z=H)=0 \tag{6.46}$$

$$\vec{n}.\nabla T_1(r=1)=\frac{dT_1}{dr}(r=1)=0 \tag{6.47}$$

$$\vec{n}.\nabla T_1(r=R_{out})=\frac{dT_1}{dr}(r=R_{out})=0 \tag{6.48}$$

$$\vec{n}\cdot\vec{V}_1(r=1)=V_{r_1}(r=1)=0 \tag{6.49}$$

and

$$\vec{n}\cdot\vec{V}_1(r=R_{out})=V_{r_1}(r=R_{out})=0 \tag{6.50}$$

Notice that in this problem we have many variables (*i.e.*, $T_1, \vec{V}_1$ and $\mathbb{P}_1$), and to simplify our calculation we need to reduce the number of these variables. One way to do this is to take the curl of Equation (6.41), namely

$$\nabla\times\vec{V}_1 = -\nabla\times\left(\nabla\mathbb{P}_1\right) - Ra\nabla\times\left(T_1\vec{i}_z\right) = -Ra\nabla T_1\times\vec{i}_z \tag{6.51}$$

or

$$\frac{1}{r}\left[\left(\frac{\partial V_{z_1}}{\partial\theta}-r\frac{\partial V_{\theta_1}}{\partial z}\right)\vec{i}_r - r\left(\frac{\partial V_{z_1}}{\partial r}-\frac{\partial V_{r_1}}{\partial z}\right)\vec{i}_\theta + \left(\frac{\partial\left(rV_{\theta_1}\right)}{\partial r}-\frac{\partial V_{r_1}}{\partial\theta}\right)\vec{i}_z\right] = -\frac{Ra}{r}\left[\frac{\partial T_1}{\partial\theta}\vec{i}_r - r\frac{\partial T_1}{\partial r}\vec{i}_\theta\right] \tag{6.52}$$

Now, let us put Equation (6.52) in a simpler form by combining it with the equation of continuity, which means

$$\frac{1}{r}\frac{\partial\left(rV_{r_1}\right)}{\partial r}+\frac{1}{r}\frac{\partial V_{\theta_1}}{\partial \theta}+\frac{\partial V_{z_1}}{\partial z}=0 \tag{6.53}$$

To do this, take the dot product of Equation (6.52) first by $\vec{i}_r$, then by $\vec{i}_\theta$ and finally by $\vec{i}_z$ to get

$$\frac{1}{r}\frac{\partial V_{z_1}}{\partial \theta}-\frac{\partial V_{\theta_1}}{\partial z}=-\frac{Ra}{r}\frac{\partial T_1}{\partial \theta} \tag{6.54}$$

$$-\frac{\partial V_{z_1}}{\partial r}+\frac{\partial V_{r_1}}{\partial z}=Ra\frac{\partial T_1}{\partial r} \tag{6.55}$$

and

$$\frac{1}{r}\frac{\partial\left(rV_{\theta_1}\right)}{\partial r}-\frac{1}{r}\frac{\partial V_{r_1}}{\partial \theta}=0 \tag{6.56}$$

Now, multiply Equation (6.55) by r, viz.,

$$-r\frac{\partial V_{z_1}}{\partial r}+\frac{\partial\left(rV_{r_1}\right)}{\partial z}=Ra\ r\frac{\partial T_1}{\partial r} \tag{6.57}$$

and then differentiate with respect to r and multiply by $(1/r)$ to get

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_{z_1}}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\left(rV_{r_1}\right)}{\partial r}\right)=Ra\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1}{\partial r}\right) \tag{6.58}$$

Now using

$$\frac{1}{r}\frac{\partial\left(rV_{r_1}\right)}{\partial r}=-\left(\frac{1}{r}\frac{\partial V_{\theta_1}}{\partial \theta}+\frac{\partial V_{z_1}}{\partial z}\right) \tag{6.59}$$

and

$$\frac{\partial V_{\theta_1}}{\partial z} = \frac{Ra}{r}\frac{\partial T_1}{\partial \theta} + \frac{1}{r}\frac{\partial V_{z_1}}{\partial \theta} \tag{6.60}$$

Equation (6.58) will become

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_{z_1}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V_{z_1}}{\partial \theta^2} + \frac{\partial^2 V_{z_1}}{\partial z^2} = -Ra\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T_1}{\partial \theta^2}\right] \tag{6.61}$$

or

$$\nabla^2 V_{z_1} = -Ra\nabla^2_{r,\theta} T_1 \tag{6.62}$$

Now, because T_o depends only on z, Equation (6.40) is just

$$\frac{\partial T_1}{\partial t} + V_{z_1}\frac{\partial T_o}{\partial z} = \nabla^2 T_1 + \delta^2 T_1 \tag{6.63}$$

and we have two equations in T_1 and V_{z_1}. It remains only to restate the boundary conditions in terms of V_{z_1}. At the surface of the rod (*i.e.*, at $r = 1$), we can write

$$\vec{n}\cdot\vec{V}_1 = V_{r_1} = 0 \quad , \quad \text{on-center} \tag{6.64}$$

and hence, $\left(dV_{r_1}/dz\right) = 0$ at $r = 1$. Then by using Equations (6.47) and (6.55) we have

$$\frac{dV_{z_1}}{dr} = 0 \ , \quad \text{at } r = 1 \tag{6.65}$$

and likewise

$$\frac{dV_{z_1}}{dr} = 0 \ , \quad \text{at } r = R_{out} \tag{6.66}$$

Our problem then is

$$\frac{\partial T_1}{\partial t} + V_{z_1}\frac{\partial T_o}{\partial z} = \nabla^2 T_1 + \delta^2 T_1$$

and

$$\nabla^2 V_{z_1} = -Ra\nabla^2_{r,\theta} T_1$$

on the domain and

$$T_1\left(z=0\right)=0 \tag{6.67}$$

$$T_1\left(z=H\right)=0 \tag{6.68}$$

$$V_{z_1}\left(z=0\right)=0 \tag{6.69}$$

$$V_{z_1}\left(z=H\right)=0 \tag{6.70}$$

$$\frac{dT_1}{dr}\left(r=1\right)=0 \tag{6.71}$$

$$\frac{dT_1}{dr}\left(r=R_{out}\right)=0 \tag{6.72}$$

$$\frac{dV_{z_1}}{dr}\left(r=1\right)=0 \tag{6.73}$$

and

$$\frac{dV_{z_1}}{dr}\left(r=R_{out}\right)=0 \tag{6.74}$$

This problem has solutions of the form

$$T_1=\hat{T}_1\left(z\right)e^{\sigma t}e^{im\theta}f\left(r\right) \tag{6.75}$$

and

$$V_{z_1}=\hat{V}_{z_1}\left(z\right)e^{\sigma t}e^{im\theta}f\left(r\right) \tag{6.76}$$

whence $\hat{T}_1\left(z\right)$, $\hat{V}_{z_1}\left(z\right)$ and $f\left(r\right)$ must satisfy

$$\sigma\hat{T}_1 f+\hat{V}_{z_1}f\frac{\partial T_o}{\partial z}=\hat{T}_1\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)-\hat{T}_1 f\frac{m^2}{r^2}+f\frac{\partial^2\hat{T}_1}{\partial z^2}+\delta^2\hat{T}_1 f \tag{6.77}$$

and

$$\hat{V}_{z_1}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)-\hat{V}_{z_1}f\frac{m^2}{r^2}+f\frac{\partial^2\hat{V}_{z_1}}{\partial z^2}=-Ra\left(\hat{T}_1\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)-\hat{T}_1 f\frac{m^2}{r^2}\right) \tag{6.78}$$

and upon setting

$$\frac{1}{f}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)-\frac{m^2}{r^2}=-k^2 \tag{6.79}$$

where $(df/dr)=0$ at $r=1$ and $r=R_{out}$, we have

$$\sigma\hat{T}_1+\hat{V}_{z_1}\frac{dT_o}{dz}=-k^2\hat{T}_1+\frac{d^2\hat{T}_1}{dz^2}+\delta^2\hat{T}_1 \tag{6.80}$$

and

$$-k^2\hat{V}_{z_1}+\frac{d^2\hat{V}_{z_1}}{dz^2}=Ra\hat{T}_1k^2 \tag{6.81}$$

This is an eigenvalue problem. The output variable is σ, the growth constant. The input variables are k^2, δ^2 and Ra. At neutral conditions $\sigma=0$ and our job is to find Ra as a function of k^2 and δ^2. To do this we must solve

$$\left[\frac{d^2}{dz^2}-k^2\right]\hat{T}_1+\delta^2\hat{T}_1=\hat{V}_{z_1}\frac{dT_o}{dz} \tag{6.82}$$

and

$$\left[\frac{d^2}{dz^2}-k^2\right]\hat{V}_{z_1}=Rak^2\hat{T}_1 \tag{6.83}$$

where

$$\hat{T}_1(z=0)=0 \tag{6.84}$$

$$\hat{T}_1(z=H)=0 \tag{6.85}$$

$$\hat{V}_{z_1}(z=0)=0 \tag{6.86}$$

$$\hat{V}_{z_1}(z=H)=0 \tag{6.87}$$

and we hold δ^2 fixed at any value that leads to a first base solution.

Eliminating $\hat{T}_1$ in favor of $\hat{V}_{z_1}$ we get

$$\left[\frac{d^2}{dz^2}-k^2\right]^2\hat{V}_{z_1}+\delta^2\left[\frac{d^2}{dz^2}-k^2\right]\hat{V}_{z_1}-Rak^2\frac{dT_o}{dz}\hat{V}_{z_1}=0 \tag{6.88}$$

where at the top and the bottom

$$\hat{V}_{z_1}=0=\left[\frac{d^2}{dz^2}-k^2\right]\hat{V}_{z_1}$$

and where $\left(dT_o/dz\right)$ is not a constant but a linear function of z.

Now $\sin\frac{n\pi}{H}z$ satisfies all four boundary conditions for any positive integer value of n. Hence we can get a solution in the form

$$\hat{V}_{z_1}=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi}{H}z \tag{6.89}$$

where the constants $B_1, B_2, \ldots$ must be determined simultaneously. But we can get an approximation by truncation as explained in Appendix E.

Let us begin with

$$\hat{V}_{z_1}=B_1\sin\frac{\pi}{H}z+B_2\sin\frac{2\pi}{H}z \tag{6.90}$$

and substitute this two-term expansion into Equation (6.88) to get

$$\begin{aligned}
&\frac{\pi^4}{H^4}\left(B_1\sin\frac{\pi}{H}z+16B_2\sin\frac{2\pi}{H}z\right)+2k^2\frac{\pi^2}{H^2}\left(B_1\sin\frac{\pi}{H}z\right.\\
&\left.+4B_2\sin\frac{2\pi}{H}z\right)+k^4\left(B_1\sin\frac{\pi}{H}z+B_2\sin\frac{2\pi}{H}z\right)\\
&-\delta^2\left[\frac{\pi^2}{H^2}\left(B_1\sin\frac{\pi}{H}z+4B_2\sin\frac{2\pi}{H}z\right)+k^2\left(B_1\sin\frac{\pi}{H}z\right.\right.\\
&\qquad\left.\left.+B_2\sin\frac{2\pi}{H}z\right)\right]-Rak^2\delta\frac{\left(-\sin\delta z\sin\delta H+\left(1-\cos\delta H\right)\cos\delta z\right)}{\sin\delta H}\\
&\qquad\qquad\times\left(B_1\sin\frac{\pi}{H}z+B_2\sin\frac{2\pi}{H}z\right)=0
\end{aligned}\tag{6.91}$$

Then, multiply this first by $\sin\frac{\pi}{H}z$, second by $\sin\frac{2\pi}{H}z$ and integrate over $(0,H)$ to get

$$\frac{\left(k^2H^2+\pi^2\right)\left(k^2H^2-\delta^2H^2+\pi^2\right)}{2H^3}B_1+\frac{8\pi^2\delta^2H^2k^2Ra}{\left(\delta^2H^2-9\pi^2\right)\left(\delta^2H^2-\pi^2\right)}B_2=0 \tag{6.92}$$

and

$$\frac{8\pi^2\delta^2H^2k^2Ra}{\left(\delta^2H^2-9\pi^2\right)\left(\delta^2H^2-\pi^2\right)}B_1+\frac{\left(k^2H^2+4\pi^2\right)\left(k^2H^2-\delta^2H^2+4\pi^2\right)}{2H^3}B_2=0 \tag{6.93}$$

This is a set of two linear homogeneous equations in B_1 and B_2. To find a solution other than $B_1=0$ and $B_2=0$ the determinant of the coefficients must vanish and this determines the critical value of Ra in this two-term approximation.

The result is

$$\begin{aligned}
Ra_{critical}&=\frac{\pi^4}{16H^3M^2\delta^2}\left(\left(\frac{\delta H}{\pi}\right)^2-9\right)\left(\left(\frac{\delta H}{\pi}\right)^2-1\right)\\
&\quad\times\left(\left(M^2+1\right)\left(M^2-\left(\frac{\delta H}{\pi}\right)^2+1\right)\left(M^2+4\right)\left(M^2-\left(\frac{\delta H}{\pi}\right)^2+4\right)\right)^{\frac{1}{2}}
\end{aligned}\tag{6.94}$$

where $M^2 = (kH/\pi)^2$. This gives $Ra_{critical}$ in terms of H, δ^2 and k^2 but k^2 remains to be determined. It is important to mention that to do this calculation we used four and even six terms expansions but we found that the minimum $Ra_{critical}$ is always the same. Based on this we learned that two-term expansion is enough.

To find k^2 we turn to the problem of f (*i.e.*, Equation (6.79)), viz.,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\left(k^2-\frac{m^2}{r^2}\right)f=0 \tag{6.95}$$

where $(df/dr)=0$ at $r=1$ and $r=R_{out}$. At every value of m, $m=0,1,2,\ldots$ this problem determines infinitely many values of k^2. But it would appear from Equation (6.94) that we will be interested in the least value of all these k^2's and this occurs at $m=0$. So we focus on $m=0$.

Hence, f can be taken to be a linear combination of $J_o(kr)$ and $Y_o(kr)$ whereupon k must be a root of

$$\left.\frac{dJ_o(kr)}{dr}\right|_{r=1}\left.\frac{dY_o(kr)}{dr}\right|_{r=R_{out}}-\left.\frac{dJ_o(kr)}{dr}\right|_{r=R_{out}}\left.\frac{dY_o(kr)}{dr}\right|_{r=1}=0 \tag{6.96}$$

in order that the linear combination not vanish everywhere. The roots of this equation determine the value of k for any value of R_{out}. The smallest values of k's at some different values of R_{out} are presented in Table 6-1.

Table 6-1. Smallest values of k at $m=0$ and various values of R_{out}.

R_{out}	k
1.5	6.321
2.5	2.073
5.0	0.763

The corresponding eigenfunction is given by

$$f = C\left[-J_o(kr)\left.\frac{dY_o(kr)}{dr}\right|_{r=1} + \left.\frac{dJ_o(kr)}{dr}\right|_{r=1} Y_o(kr)\right] \tag{6.97}$$

where C is an arbitrary constant. Now, T_1 and V_{z_1}, at $\sigma = 0$ and $m = 0$, given k^2 and δ^2, can be written as

$$\begin{aligned} T_1 = \Bigg(\frac{C}{Rak^2}\left[-\frac{\pi^2}{H^2}B_1\sin\frac{\pi}{H}z - \frac{4\pi^2}{H^2}B_2\sin\frac{2\pi}{H}z\right] - k^2 B_1 \sin\frac{\pi}{H}z \\ + B_2\sin\frac{2\pi}{H}z\Bigg) \times \left(-J_o(kr)\left.\frac{dY_o(kr)}{dr}\right|_{r=1} + \left.\frac{dJ_o(kr)}{dr}\right|_{r=1} Y_o(kr)\right) \end{aligned} \tag{6.98}$$

and

$$V_{z_1} = C\left(B_1\sin\frac{\pi}{H}z + B_2\sin\frac{2\pi}{H}z\right)\left(-J_o(kr)\left.\frac{dY_o(kr)}{dr}\right|_{r=1} + \left.\frac{dJ_o(kr)}{dr}\right|_{r=1} Y_o(kr)\right) \tag{6.99}$$

where k^2, H^2 and δ^2 are independent. k^2 depends on R_{out}. The critical Rayleigh number as a function of δ^2 and at $R_{out} = 1.5$, $H = 1$ is illustrated in Figure 6-2.

At $\delta^2 = 0$ we have $Ra_{critical} = \infty$. As δ^2 increases the porous medium heats up and density gradients become steeper. As a result $Ra_{critical}$ falls and reaches zero when δ^2 reaches its no flow critical value where the temperature becomes unbounded.

We have reached a point where we can see what the problem really is if the value of Ra is set and δ^2 is increased, then what is the critical value of δ^2. We expect it to be higher than $\delta^2_{critical}$ at no flow and our earlier work with flow would suggest this. But we cannot do the calculation and in particular we cannot hope to do it when the pipe is displaced. So we turn our attention to things we can do.

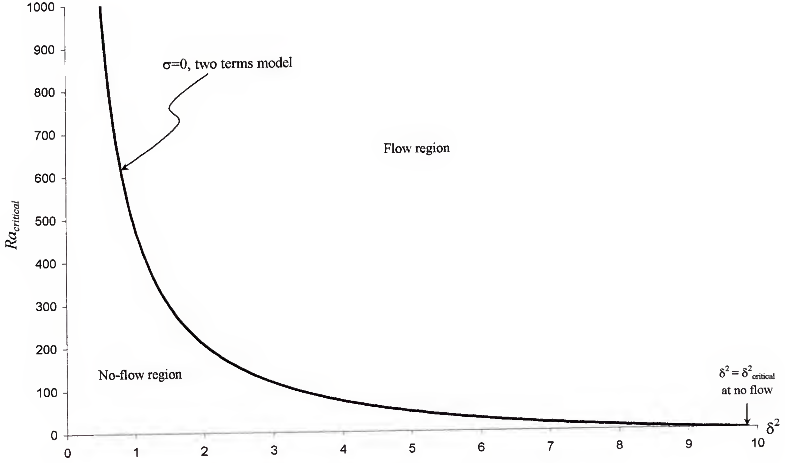


Figure 6-2. Critical value of Ra at $R_{out} = 1.5$, $H = 1$ and various δ^2.

Also, holding δ^2 fixed $Ra_{critical}$ *vs.* k for fixed R_{out} or $Ra_{critical}$ *vs.* R_{out} for a fixed k would be a parabolic shaped curve starting at infinity and returning to infinity. But k is a discrete variable at a given R_{out} and each R_{out} determines its own set of k 's. It is interesting then to hold the root number fixed and plots $Ra_{critical}$ *vs.* R_{out}.

Now, let us hold δ^2 fixed and plot the critical value of Ra *vs.* R_{out}, first for the least root, then for the next least root, etc. Denoting these roots $n = 1, 2, \ldots$ and remembering that the roots depend on R_{out}, this graph is illustrated in Figure 6-3.

Now we have to be careful. The roots at $m = 1$, $m = 2$, etc interweaves with those at $m = 0$. So while the lowest is at $m = 0$ the next lowest is at $m = 1$.

It is interesting to notice that at $R_{out} = 2$ the value of $Ra_{critical}$ is 465 and it corresponds to $n = 1$, the first root. But if $R_{out} = 4$, $Ra_{critical} = 427$ and this corresponds to $n = 4$, the fourth root. For small values of R_{out} it is difficult to start natural convection. But on increasing R_{out} we come to a point where the critical Rayleigh number fluctuates in a narrow range. What appears to happen is that the system more easily accommodates one cell, then two cells, then three cells, etc corresponds to $n = 1, 2, \ldots$. To make this clear we can plot the temperature and the velocity pattern for $R_{out} = 2$ and $R_{out} = 4$. These are shown in Figures 6-4 and 6-5.

There is an interval of R_{out} where, say, n cells become easier to accommodate from $n-1$ cells, then harder to accommodate, giving way to $n+1$ cells.

Now we can go on, at fixed δ^2, and displace the rod to see what effect this has on the structure of the flow. In doing this there is really no need to carry along the

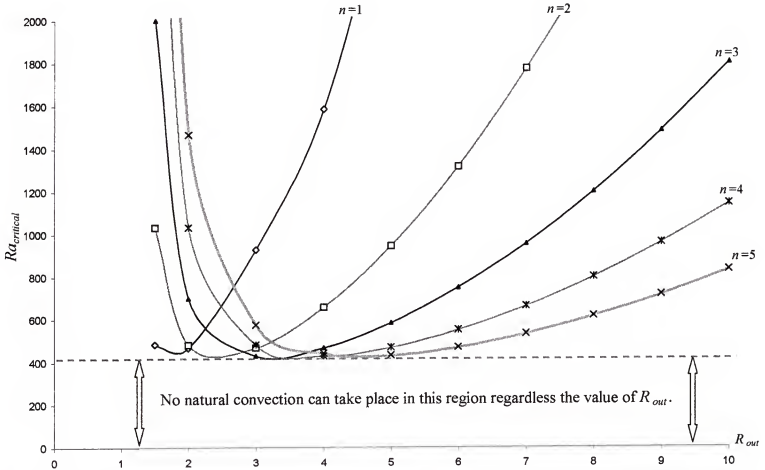
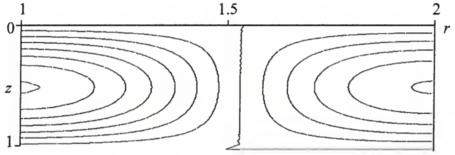
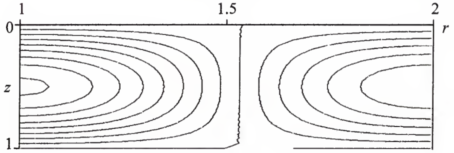


Figure 6-3. Critical values of Ra at $\delta^2 = 1$, $H = 1$, $m = 0$ and various values of R_{out}. The values of k depend on R_{out} and are denoted $n = 1, 2, \ldots$.

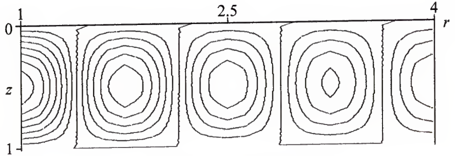


Temperature pattern

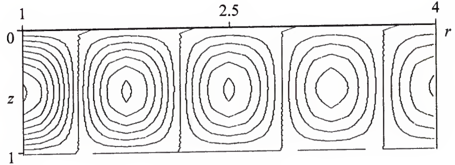


Velocity pattern

Figure 6-4. Temperature and the velocity patterns at $\delta^2 = 1$, $H = 1$, $R_{out} = 2$, $Ra = 465$ and $k = 3.2$.



Temperature pattern



Velocity pattern

Figure 6-5. Temperature and the velocity pattern at $\delta^2 = 1$, $H = 1$, $R_{out} = 4$, $Ra = 427$ and $k = 4.21$.

complication of a temperature dependent heat source, as we are not trying to solve a thermal explosion problem. So, to simplify our work we take the heat source to be constant. This does not simplify the base solution much, but it does simplify the perturbation problem due to the displacement of the rod. Before starting it may be worth noticing that the two perturbations will be carried out, one specified, one not. So we need to be careful about the meaning of ϵ.

Now, by setting Q equal to a constant there is no question of thermal stability and our focus becomes entirely on natural convection. The scaling is now a little different. Taking the scale factors to be $\overline{T} = QR_{in}^2/k_c$, $\overline{V} = k_c/\rho C_v R_{in}$, $\overline{\nabla} = 1/R_{in}$, $\overline{\mathbb{P}} = \mu k_c/k_p \rho C_v$ and $\overline{t} = \rho C_v R_{in}^2/k_c$. By dropping the *'s on the scaled variables, the equations to be solved are

$$\frac{\partial T}{\partial t} + \vec{V} \cdot \nabla T = \nabla^2 T + 1 \tag{6.100}$$

$$\vec{V} = -\nabla \mathbb{P} - RaT\vec{i}_z \tag{6.101}$$

and

$$\nabla \cdot \vec{V} = 0 \tag{6.102}$$

where

$$Ra = \frac{\alpha k_p R_{in}^3 Q g_z}{k_d \nu k_c}$$

while the boundary conditions remain unchanged and they are

$$T(z=0) = 0 \tag{6.103}$$

$$T(z=H) = 0 \tag{6.104}$$

$$V_z(z=0) = 0 \tag{6.105}$$

$$V_z(z=H)=0 \tag{6.106}$$

$$\vec{n}\cdot\nabla T(r=1)=0 \tag{6.107}$$

$$\vec{n}\cdot\nabla T(r=R_{out})=0 \tag{6.108}$$

$$\vec{n}\cdot\vec{V}(r=1)=0 \tag{6.109}$$

and

$$\vec{n}\cdot\vec{V}(r=R_{out})=0 \tag{6.110}$$

Again, notice that Ra is not a control variable, it depends on the properties of the porous medium, the fluid and the geometry. The value of Ra will increase as the heat source becomes stronger (strength the density gradient), the permeability increases (fluid flows easily through big gaps), the thermal expansion increases (strength the density gradient), the thermal diffusivity decreases (hard to eliminate the horizontal temperature perturbation), the thermal conductivity decreases (difficult to remove the heat), kinematic viscosity decreases (less resistance to flow) and as the gravity increases (strength the density gradient).

Let us begin by presenting the base solution for this problem. It is found to be

$$T_o=\frac{z}{2}(H-z) \tag{6.111}$$

and

$$\vec{V}_o=0 \tag{6.112}$$

and by this, there obtains

$$\frac{dT_o}{dz}=\left(\frac{H}{2}-z\right) \tag{6.113}$$

which will be needed later. Given T_o, $\mathbb{P}_o$ can be obtained but it is not needed. Again the base solution holds for any Ra.

The base solution is the same no matter where the pipe is located. The first order problem obtained by introducing an upset into the base solution and used to determine the critical value of Ra, again for all pipe positions, will be

$$\frac{\partial T_1}{\partial t}+\vec{V}_1\cdot\nabla T_o=\nabla^2 T_1 \tag{6.114}$$

$$\vec{V}_1=-\nabla\mathbb{P}_1-RaT_1\vec{i}_z \tag{6.115}$$

and

$$\nabla\cdot\vec{V}_1=0 \tag{6.116}$$

where the boundary conditions to be satisfied are

$$T_1(z=0)=0\ ,\quad \text{at the top wall} \tag{6.117}$$

$$T_1(z=H)=0\ ,\quad \text{at the bottom wall} \tag{6.118}$$

$$V_{z_1}(z=0)=0\ ,\quad \text{at the top wall} \tag{6.119}$$

$$V_{z_1}(z=H)=0\ ,\quad \text{at the bottom wall} \tag{6.120}$$

$$\vec{n}.\nabla T_1(r=R(\theta,\varepsilon))=0\ ,\quad \text{at the rod wall} \tag{6.121}$$

$$\vec{n}.\nabla T_1(r=R_{out})=\frac{dT_1}{dr}(r=R_{out})=0\ ,\quad \text{at the pipe wall} \tag{6.122}$$

$$\vec{n}\cdot\vec{V}_1(r=R(\theta,\varepsilon))=0\ ,\quad \text{at the rod wall} \tag{6.123}$$

and

$$\vec{n}\cdot\vec{V}_1(r=R_{out})=V_{r_1}(r=R_{out})=0\ ,\quad \text{at the pipe wall} \tag{6.124}$$

As before the domain equations reduce to two equations in T_1 and V_{z_1}. They are

$$\frac{\partial T_1}{\partial t} + V_{z_1}\frac{\partial T_o}{\partial z} = \nabla^2 T_1 \tag{6.125}$$

and

$$\nabla^2 V_{z_1} = -Ra\nabla^2_{r,\theta} T_1 \tag{6.126}$$

where

$$T_1(z=0)=0\ , \quad \text{at the top wall} \tag{6.127}$$

$$T_1(z=H)=0\ , \quad \text{at the bottom wall} \tag{6.128}$$

$$V_{z_1}(z=0)=0\ , \quad \text{at the top wall} \tag{6.129}$$

$$V_{z_1}(z=H)=0\ , \quad \text{at the bottom wall} \tag{6.130}$$

$$\vec{n}.\nabla T_1(r=R(\theta,\varepsilon))=0\ , \quad \text{at the rod wall} \tag{6.131}$$

$$\vec{n}.\nabla T_1(r=R_{out})=\frac{dT_1}{dr}(r=R_{out})=0\ , \quad \text{at the pipe wall} \tag{6.132}$$

$$\vec{n}\cdot\nabla V_{z_1}(r=R(\theta,\varepsilon))=0\ , \quad \text{at the rod wall} \tag{6.133}$$

and

$$\vec{n}\cdot\nabla V_{z_1}(r=R_{out})=\frac{dV_{z_1}}{dr}(r=R_{out})=0\ , \quad \text{at the pipe wall} \tag{6.134}$$

Now we write the eigenvalue problem as a base eigenvalue problem corresponding to the on-center case and then the first, the second, etc problems on the reference domain corresponding to moving the rod off center.

The base eigenvalue problem is as above with $dV_{z_1}/dr=0$ and $dT_1/dr=0$ at $r=1$.

To solve the base problem write

$$T_1 = \hat{T}_1(z)e^{\sigma t}e^{im\theta}f_o(r) \tag{6.135}$$

and

$$V_{z_1} = \hat{V}_{z_1}(z)e^{\sigma t}e^{im\theta}f_o(r) \tag{6.136}$$

whence setting

$$\frac{1}{f_o}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f_o}{\partial r}\right) - \frac{m^2}{r^2} = -k^2 \tag{6.137}$$

where $(df/dr) = 0$ at $r = 1$ and $r = R_{out}$, $\hat{T}_1$ and $\hat{V}_{z_1}$ must satisfy

$$\sigma\hat{T}_1 + \hat{V}_{z_1}\frac{dT_o}{dz} = -k^2\hat{T}_1 + \frac{d^2\hat{T}_1}{dz^2} \tag{6.138}$$

and

$$-k^2\hat{V}_{z_1} + \frac{d^2\hat{V}_{z_1}}{dz^2} = Ra\hat{T}_1k^2 \tag{6.139}$$

on the reference domain while at the boundaries

$$\hat{T}_1(z=0) = 0 \tag{6.140}$$

$$\hat{T}_1(z=H) = 0 \tag{6.141}$$

$$\hat{V}_{z_1}(z=0) = 0 \tag{6.142}$$

and

$$\hat{V}_{z_1}(z=H) = 0 \tag{6.143}$$

To solve this problem for the critical value of Ra as a function of k^2 set $\sigma = 0$ and eliminate $\hat{T}_1$ in favor of $\hat{V}_{z_1}$ to get

$$\left[\frac{d^2}{dz^2} - k^2\right]^2 \hat{V}_{z_1} - Rak^2\frac{dT_o}{dz}\hat{V}_{z_1} = 0 \tag{6.144}$$

where

$$\frac{dT_o}{dz} = \frac{H}{2} - z$$

and where $\hat{V}_{z_1}$ at the top and the bottom satisfies

$$\hat{V}_{z_1} = 0 = \left[\frac{d^2}{dz^2} - k^2\right]\hat{V}_{z_1}$$

As before $\hat{V}_{z_1}$ can be found in a series which satisfy all the boundary conditions, term by term, viz.,

$$\hat{V}_{z_1} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{H}z \tag{6.145}$$

Calculating this off at $n = 2$ and substituting it into Equation (6.144) results in

$$\begin{aligned}
&\frac{\pi^4}{H^4}\left(B_1 \sin\frac{\pi}{H}z + 16B_2 \sin\frac{2\pi}{H}z\right) + 2k^2\frac{\pi^2}{H^2}\left(B_1 \sin\frac{\pi}{H}z\right.\\
&\left.+4B_2 \sin\frac{2\pi}{H}z\right) + k^4\left(B_1 \sin\frac{\pi}{H}z + B_2 \sin\frac{2\pi}{H}z\right)\\
&-Rak^2\left(B_1 \sin\frac{\pi}{H}z + B_2 \sin\frac{2\pi}{H}z\right)\left(\frac{H}{2} - z\right) = 0
\end{aligned} \tag{6.146}$$

Then, multiply Equation (6.146) by $\sin\frac{\pi}{H}z$ and integrate with respect to z over $(0, H)$ to get

$$\frac{\left(k^2H^2 + \pi^2\right)^2}{2H^3}B_1 - \frac{8H^2k^2Ra}{9\pi^2}B_2 = 0 \tag{6.147}$$

and multiply Equation (6.146) by $\sin\frac{2\pi}{H}z$ and integrate with respect to z over $(0, H)$ to get

$$-\frac{8H^2k^2Ra}{9\pi^2}B_1 + \frac{\left(k^2H^2 + 4\pi^2\right)^2}{2H^3}B_2 = 0 \tag{6.148}$$

To find a solution other than $B_1 = 0$ and $B_2 = 0$, the determinant of the coefficients of B_1 and B_2 must vanish. This gives the critical value of Ra as

$$Ra_{critical} = \frac{9\pi^4 \left(M^2+1\right)\left(M^2+4\right)}{16H^3M^2} \tag{6.149}$$

where $M = \left(kH/\pi\right)$.

Notice that $Ra_{critical}$ depends on M^2 and H. M^2 depends on k^2 which itself depends on R_{out} and m. It is important to mention that to do this calculation we used four and even six terms expansions but we found that the minimum $Ra_{critical}$ is always the same. Based on this we learned that two-term expansion is enough.

Now given R_{out} and m, f_o and the allowable values of k can be determined just like before. Then, T_1 and V_{z_1} at $m = 0$, can be written as

$$\begin{aligned} T_1 = &\left(\frac{C}{Rak^2}\left[-\frac{\pi^2}{H^2}B_1\sin\frac{\pi}{H}z - \frac{4\pi^2}{H^2}B_2\sin\frac{2\pi}{H}z\right] - k^2B_1\sin\frac{\pi}{H}z\right. \\ &\left.+B_2\sin\frac{2\pi}{H}z\right)\times\left(-J_o\left(kr\right)\left.\frac{dY_o\left(kr\right)}{dr}\right|_{r=1} + \left.\frac{dJ_o\left(kr\right)}{dr}\right|_{r=1} Y_o\left(kr\right)\right) \end{aligned} \tag{6.150}$$

and

$$\begin{aligned} V_{z_1} = C&\left(B_1\sin\frac{\pi}{H}z + B_2\sin\frac{2\pi}{H}z\right) \\ &\times\left(-J_o\left(kr\right)\left.\frac{dY_o\left(kr\right)}{dr}\right|_{r=1} + \left.\frac{dJ_o\left(kr\right)}{dr}\right|_{r=1} Y_o\left(kr\right)\right) \end{aligned} \tag{6.151}$$

Without making a choice of R_{out} or m, we can plot $Ra_{critical}$ *vs.* M^2 at fixed values of H. Such a graph is illustrated in Figure 6-6. In this figure we observe that the critical value of Ra goes to infinity as M^2 goes to zero or infinity (*i.e.*, as the horizontal

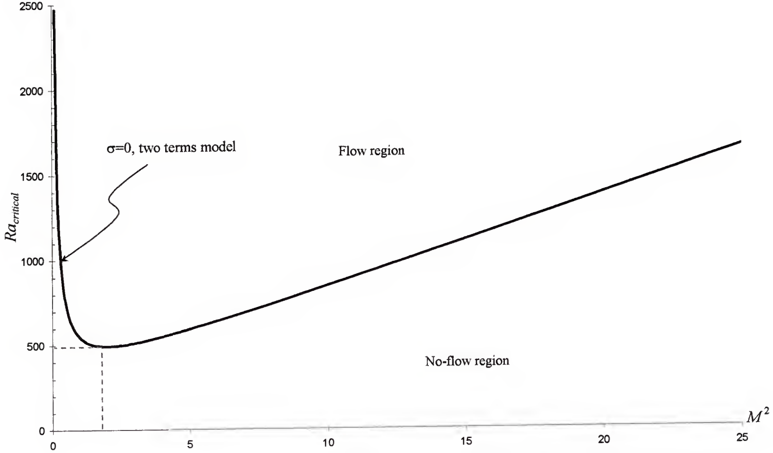


Figure 6-6. Critical values of Ra at $H=1$ and various values of M^2. ($Q=\text{Constant}$)

variation of T_1 becomes weaker or stronger), the first wiping out horizontal density variations, the second strengthening horizontal heat conduction. Both effects make the onset of flow more difficult to achieve.

We might like to make a comparison between this and the earlier case. The difference lies in the base solution and there we had a δ^2, here we do not.

Now the values of m and R_{out} determines the allowable values of k as the solution to Equation (6.137). In each case there are many roots and we denote these roots by n, $n = 1, 2, \ldots$, $n = 1$ corresponding to the least k at a given m and R_{out}. We can set m, here $m = 0$, and draw a graph of $Ra_{critical}$ *vs.* R_{out} for $n = 1$, $n = 2$, …, the lowest the next lowest, etc root at each R_{out}. This is shown in Figure 6-7.

Again, we have to be careful, the roots at $m = 1$, $m = 2$, etc interweave those at $m = 0$. So while the lowest is at $m = 0$ the next lowest is at $m = 1$.

It is interesting to notice that at $R_{out} = 2$ the value of $Ra_{critical}$ is 550 and it corresponds to $n = 1$, the first root. But if $R_{out} = 4$, $Ra_{critical} = 494$ and this corresponds to $n = 4$, the fourth root. It is same as before, for small values of R_{out} it is difficult to start natural convection. But on increasing R_{out} we come to a point where the critical Rayleigh number fluctuates in a narrow range. Again, this is happens because increasing R_{out} will increase number of flow cells and this will lower the value of $Ra_{critical}$.

Now, let us return to our main interest and try to learn the effect of displacing the rod. This has no effect on the base problem; hence the eigenvalue problem is our focus. So we have a base eigenvalue problem, solved above, a first order correction, a second

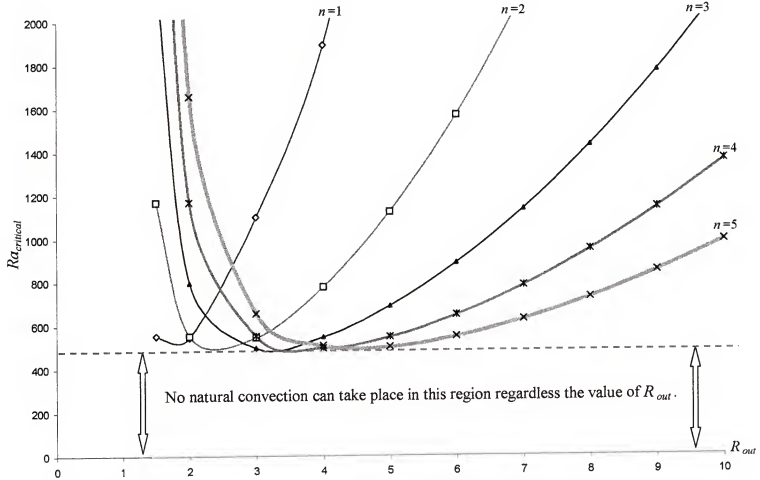


Figure 6-7. Critical values of Ra at $H=1$ and various values of R_{out} and at different levels of k. ($Q=$ Constant)

order correction, etc in power of ε, where ε measures the displacement of the rod. The off-center problem is given by equation

$$\nabla^2 T_1 - V_{z_1}\frac{\partial T_o}{\partial z} = 0 \tag{6.152}$$

and

$$\nabla^2 V_{z_1} = -Ra\nabla^2_{r,\theta} T_1 \tag{6.153}$$

and the boundary conditions to be satisfied are

$$T_1(z=0)=0 \tag{6.154}$$

$$T_1(z=H)=0 \tag{6.155}$$

$$V_{z_1}(z=0)=0 \tag{6.156}$$

$$V_{z_1}(z=H)=0 \tag{6.157}$$

$$\vec{n}.\nabla T_1(r=R(\theta,\varepsilon))=0 \tag{6.158}$$

$$\vec{n}.\nabla T_1(r=R_{out})=0 \tag{6.159}$$

$$\vec{n}\cdot\nabla V_{z_1}(r=R(\theta,\varepsilon))=0 \tag{6.160}$$

and

$$\vec{n}\cdot\nabla V_{z_1}(r=R_{out})=0 \tag{6.161}$$

and the on-centered problem (*i.e.*, at $\varepsilon = 0$), has been solved earlier, whence T_1, V_{z_1} and Ra there are T_{1_o}, $V_{z_{1_o}}$ and Ra_o here.

The problems at order ε^1 and order ε^2 will now be written by methods familiar from earlier chapters. They hold on the reference domain and they are[1] at order ε^1:

$$\nabla^2 T_{1_1} - V_{z_{1_1}} \frac{dT_o}{dz} = 0 \tag{6.162}$$

and

$$\nabla^2 V_{z_{1_1}} + Ra_o \nabla^2_{r_o,\theta_o} T_{1_1} + Ra_1 \nabla^2_{r_o,\theta_o} T_{1_o} = 0 \tag{6.163}$$

where at the boundaries T_{1_1} and $V_{z_{1_1}}$ must satisfy

$$T_{1_1}(z=0) = 0 \tag{6.164}$$

$$T_{1_1}(z=H) = 0 \tag{6.165}$$

$$V_{z_{1_1}}(z=0) = 0 \tag{6.166}$$

$$V_{z_{1_1}}(z=H) = 0 \tag{6.167}$$

$$\frac{dT_{1_1}}{dr_o}(r_o = 1) = -R_1 \frac{d^2 T_{1_o}}{dr_o^2} \tag{6.168}$$

$$\frac{dT_{1_1}}{dr_o}(r_o = R_{out}) = 0 \tag{6.169}$$

$$\frac{dV_{z_{1_1}}}{dr_o}(r_o = 1) = -R_1 \frac{d^2 V_{z_{1_o}}}{dr_o^2} \tag{6.170}$$

and

$$\frac{dV_{z_{1_1}}}{dr_o}(r_o = R_{out}) = 0 \tag{6.171}$$

where $R_1 = \cos\theta_o$.

---

[1] Note that the perturbed temperature boundary condition is no longer adiabatic because it tries to simulate adiabatic conditions at the displaced surface by putting a condition other than adiabatic on the reference surface.

At order ε^2:

$$\nabla^2 T_{1_2} - V_{z_{1_2}} \frac{dT_o}{dz} = 0 \tag{6.172}$$

and

$$\nabla^2 V_{z_{1_2}} + Ra_o \nabla^2_{r_o,\theta_o} T_{1_2} + 2Ra_1 \nabla^2_{r_o,\theta_o} T_{1_1} + Ra_2 \nabla^2_{r_o,\theta_o} T_{1_o} = 0 \tag{6.173}$$

where at the boundaries T_{1_2} and $V_{z_{1_2}}$ must satisfy

$$T_{1_2}\left(z=0\right)=0 \tag{6.174}$$

$$T_{1_2}\left(z=H\right)=0 \tag{6.175}$$

$$V_{z_{1_2}}\left(z=0\right)=0 \tag{6.176}$$

$$V_{z_{1_2}}\left(z=H\right)=0 \tag{6.177}$$

$$\frac{\partial T_{1_2}}{\partial r_o}\left(r_o=1\right) = -2R_1 \frac{\partial^2 T_{1_1}}{\partial r_o^2} - R_1^2 \frac{\partial^3 T_{1_o}}{\partial r_o^3} - R_2 \frac{\partial^2 T_{1_o}}{\partial r_o^2} - 2\frac{\sin\theta_o}{R_o^2}\frac{\partial T_{1_1}}{\partial \theta_o} \tag{6.178}$$

$$\frac{dT_{1_2}}{dr_o}\left(r_o=R_{out}\right)=0 \tag{6.179}$$

$$\frac{\partial V_{z_{1_2}}}{\partial r_o}\left(r_o=1\right) = -2R_1 \frac{\partial^2 V_{z_{1_1}}}{\partial r_o^2} - R_1^2 \frac{\partial^3 V_{z_{1_o}}}{\partial r_o^3} - R_2 \frac{\partial^2 V_{z_{1_o}}}{\partial r_o^2} - 2\frac{\sin\theta_o}{R_o^2}\frac{\partial V_{z_{1_1}}}{\partial \theta_o} \tag{6.180}$$

and

$$\frac{dV_{z_{1_2}}}{dr_o}\left(r_o=R_{out}\right)=0 \tag{6.181}$$

where $R_2 = -\sin^2\theta_o$.

Now, to know whether displacing the rod off-center increases the chances of starting natural convection we must determine Ra_1 and Ra_2, the corrections to the critical value of Ra at the on-center case (*i.e.*, Ra_o).

To find Ra_1, we must solve Equations (6.162) and (6.163). But before solving this problem let us see what can we learn about Ra_1. First, multiply Equation (6.162) by $V_{z_{1_o}}$, which gives

$$V_{z_{1_o}}\nabla^2 T_{1_1} - V_{z_{1_o}} V_{z_{1_1}} \frac{dT_o}{dz} = 0 \tag{6.182}$$

and multiply Equation (6.125), at steady state, by $V_{z_{1_1}}$, to get

$$V_{z_{1_1}}\nabla^2 T_{1_o} - V_{z_{1_1}} V_{z_{1_o}} \frac{dT_o}{dz} = 0 \tag{6.183}$$

Then, subtract Equation (6.183) from Equation (6.182), to obtain

$$V_{z_{1_o}}\nabla^2 T_{1_1} - V_{z_{1_1}}\nabla^2 T_{1_o} = 0 \tag{6.184}$$

Now, integrate this equation over the volume of the reference domain to get

$$\int_0^H \int_0^{2\pi} \int_1^{R_{out}} \left(V_{z_{1_o}}\nabla^2 T_{1_1} - V_{z_{1_1}}\nabla^2 T_{1_o} \right) r_o dr_o d\theta_o dz = 0 \tag{6.185}$$

This equation can be rewritten as

$$\int_0^H \int_0^{2\pi} \int_1^{R_{out}} \left(\nabla\cdot\left(V_{z_{1_o}}\nabla T_{1_1}\right) - \nabla V_{z_{1_o}}\cdot\nabla T_{1_1} - \nabla\cdot\left(V_{z_{1_1}}\nabla T_{1_o}\right) + \nabla V_{z_{1_1}}\cdot\nabla T_{1_o} \right) r_o dr_o d\theta_o dz = 0 \tag{6.186}$$

Now, use the divergence theorem to get,

$$\begin{aligned} &\int_0^H \int_0^{2\pi} \left(\vec{n}\cdot\left(V_{z_{1_o}}\nabla T_{1_1}\right)\Big|_1^{R_{out}} - \int_1^{R_{out}} \nabla V_{z_{1_o}}\cdot\nabla T_{1_1} r_o dr_o \right) d\theta_o dz \\ &\quad - \int_0^H \int_0^{2\pi} \left(\vec{n}\cdot\left(V_{z_{1_1}}\nabla T_{1_o}\right)\Big|_1^{R_{out}} - \int_1^{R_{out}} \nabla V_{z_{1_1}}\cdot\nabla T_{1_o} r_o dr_o \right) d\theta_o dz = 0 \end{aligned} \tag{6.187}$$

Then, because $V_{z_{1_o}}$ is zero at the top and the bottom and because $\vec{n}\cdot\nabla T_{1_1}$ at $r_o = 1$ is a multiple of $\cos\theta_o$ and because $\vec{n}\cdot\nabla T_{1_1}$ is zero at $r_o = R_{out}$ the first term in Equation (6.187) vanishes and because $V_{z_{1_1}}$ is zero at the top and the bottom and due to $\vec{n}\cdot\nabla T_{1_o}$ being zero at $r_o = 1$ and at $r_o = R_{out}$ the third term vanishes and the equation reduces to

$$\int_0^H \int_0^{2\pi} \int_1^{R_{out}} \left(-\nabla V_{z_{1_o}}\cdot\nabla T_{1_1} + \nabla V_{z_{1_1}}\cdot\nabla T_{1_o}\right) r_o dr_o d\theta_o dz = 0 \tag{6.188}$$

This equation will be useful later. Notice that we can get the same equation even if we begin with,

$$T_{1_o}\nabla^2 V_{z_{1_1}} - T_{1_1}\nabla^2 V_{z_{1_o}} = 0 \tag{6.189}$$

To show this, let us integrate this equation over the volume of the reference domain viz.,

$$\int_0^H \int_0^{2\pi} \int_1^{R_{out}} \left(T_{1_o}\nabla^2 V_{z_{1_1}} - T_{1_1}\nabla^2 V_{z_{1_o}}\right) r_o dr_o d\theta_o dz = 0 \tag{6.190}$$

This equation can be rewritten as

$$\int_0^H \int_0^{2\pi} \int_1^{R_{out}} \left(\nabla\cdot\left(T_{1_o}\nabla V_{z_{1_1}}\right) - \nabla T_{1_o}\cdot\nabla V_{z_{1_1}} - \nabla\cdot\left(T_{1_1}\nabla V_{z_{1_o}}\right) + \nabla T_{1_1}\cdot\nabla V_{z_{1_o}}\right) r_o dr_o d\theta_o dz = 0 \tag{6.191}$$

Now use the divergence theorem to get,

$$\begin{aligned}\int_0^H \int_0^{2\pi} \left(\vec{n}\cdot\left(T_{1_o}\nabla V_{z_{1_1}}\right)\Big|_1^{R_{out}} - \int_1^{R_{out}} \nabla T_{1_o}\cdot\nabla V_{z_{1_1}} r_o dr_o \right) d\theta_o dz \\ - \int_0^H \int_0^{2\pi} \left(\vec{n}\cdot\left(T_{1_1}\nabla V_{z_{1_o}}\right)\Big|_1^{R_{out}} - \int_1^{R_{out}} \nabla T_{1_1}\cdot\nabla V_{z_{1_o}} r_o dr_o \right) d\theta_o dz = 0\end{aligned} \tag{6.192}$$

Then, because T_{1_o} is zero at the top and the bottom and because $\vec{n}\cdot\nabla V_{z_{1_1}}$ at $r_o = 1$ is multiple of $\cos\theta_o$ and $\vec{n}\cdot\nabla V_{z_{1_1}}$ is zero at $r_o = R_{out}$ the first term in Equation (6.192) vanishes and because T_{1_1} is zero at the top and the bottom and because $\vec{n}\cdot\nabla V_{z_{1_o}}$ is zero at $r_o = 1$ and at $r_o = R_{out}$ the third term vanishes and the equation reduces to

$$\int_0^H\int_0^{2\pi}\int_1^{R_{out}}\left(-\nabla T_{1_o}\cdot\nabla V_{z_{1_1}} + \nabla T_{1_1}\cdot\nabla V_{z_{1_o}}\right)r_o dr_o d\theta_o dz = 0 \tag{6.193}$$

which is same as Equation (6.188). Therefore we can say that Equations (6.185) and (6.190) are equal.

Second, multiply Equation (6.163) by T_{1_o}, which gives

$$T_{1_o}\nabla^2 V_{z_{1_1}} + Ra_o T_{1_o}\nabla^2_{r_o,\theta_o}T_{1_1} + Ra_1 T_{1_o}\nabla^2_{r_o,\theta_o}T_{1_o} = 0 \tag{6.194}$$

and multiply Equation (6.126) by T_{1_1}, to get

$$T_{1_1}\nabla^2 V_{z_{1_o}} + Ra_o T_{1_1}\nabla^2_{r_o,\theta_o}T_{1_o} = 0 \tag{6.195}$$

Then, subtract Equation (6.195) from Equation (6.194), to obtain

$$T_{1_o}\nabla^2 V_{z_{1_1}} - T_{1_1}\nabla^2 V_{z_{1_o}} + Ra_o\left(T_{1_o}\nabla^2_{r_o,\theta_o}T_{1_1} - T_{1_1}\nabla^2_{r_o,\theta_o}T_{1_o}\right) + Ra_1 T_{1_o}\nabla^2_{r_o,\theta_o}T_{1_o} = 0 \tag{6.196}$$

Now, let us integrate Equation (6.196) over the reference domain to get,

$$\begin{aligned}\int_0^H\int_0^{2\pi}\int_1^{R_{out}}\Big(&T_{1_o}\nabla^2 V_{z_{1_1}} - T_{1_1}\nabla^2 V_{z_{1_o}}\\ &+ Ra_o\left(T_{1_o}\nabla^2_{r_o,\theta_o}T_{1_1} - T_{1_1}\nabla^2_{r_o,\theta_o}T_{1_o}\right)\\ &+ Ra_1 T_{1_o}\nabla^2_{r_o,\theta_o}T_{1_o}\Big)r_o dr_o d\theta_o dz = 0\end{aligned} \tag{6.197}$$

From Equation (6.190) we know that

$$\int_0^H\int_0^{2\pi}\int_1^{R_{out}}\left(T_{1_o}\nabla^2 V_{z_{1_1}} - T_{1_1}\nabla^2 V_{z_{1_o}}\right)r_o dr_o d\theta_o dz = 0$$

Hence, Equation (6.197) reduces to

$$\begin{aligned} Ra_o \int_0^H \int_0^{2\pi} \int_1^{R_{out}} \left(T_{1_o} \nabla^2_{r_o,\theta_o} T_{1_1} - T_{1_1} \nabla^2_{r_o,\theta_o} T_{1_o} \right) r_o dr_o d\theta_o dz \\ = -Ra_1 \int_0^H \int_0^{2\pi} \int_1^{R_{out}} T_{1_o} \nabla^2_{r_o,\theta_o} T_{1_o} r_o dr_o d\theta_o dz \end{aligned} \tag{6.198}$$

Now by using the divergence theorem we get,

$$\begin{aligned} Ra_o \Bigg[\int_0^H \int_0^{2\pi} \left(\vec{n} \cdot \left(T_{1_o} \nabla_{r_o,\theta_o} T_{1_1} \right) \Big|_1^{R_{out}} - \int_1^{R_{out}} \nabla_{r_o,\theta_o} T_{1_o} \cdot \nabla_{r_o,\theta_o} T_{1_1} r_o dr_o \right) d\theta_o dz \\ - \int_0^H \int_0^{2\pi} \left(\vec{n} \cdot \left(T_{1_1} \nabla_{r_o,\theta_o} T_{1_o} \right) \Big|_1^{R_{out}} - \int_1^{R_{out}} \nabla_{r_o,\theta_o} T_{1_o} \cdot \nabla_{r_o,\theta_o} T_{1_1} r_o dr_o \right) d\theta_o dz \Bigg] \\ = -Ra_1 \int_0^H \int_0^{2\pi} \left(\vec{n} \cdot \left(T_{1_o} \nabla_{r_o,\theta_o} T_{1_o} \right) \Big|_1^{R_{out}} - \int_1^{R_{out}} \nabla_{r_o,\theta_o} T_{1_o} \cdot \nabla_{r_o,\theta_o} T_{1_o} r_o dr_o \right) d\theta_o dz \end{aligned} \tag{6.199}$$

which can be arranged as

$$\begin{aligned} Ra_o \int_0^H \int_0^{2\pi} \left(T_{1_o} \vec{n} \cdot \nabla_{r_o,\theta_o} T_{1_1} - T_{1_1} \vec{n} \cdot \nabla_{r_o,\theta_o} T_{1_o} \right)_1^{R_{out}} d\theta_o dz \\ = -Ra_1 \int_0^H \int_0^{2\pi} \left(T_{1_o} \vec{n} \cdot \nabla_{r_o,\theta_o} T_{1_o} \Big|_1^{R_{out}} - \int_1^{R_{out}} \left(\nabla_{r_o,\theta_o} T_{1_o} \right)^2 r_o dr_o \right) d\theta_o dz \end{aligned} \tag{6.200}$$

Then, because T_{1_o} is zero at the top and the bottom and because $\vec{n} \cdot \nabla_{r_o,\theta_o} T_{1_1}$ at $r_o = 1$ is multiple of $\cos\theta_o$ and $\vec{n} \cdot \nabla_{r_o,\theta_o} T_{1_1}$ is zero at $r_o = R_{out}$ and because T_{1_1} is zero at the top and the bottom and because $\vec{n} \cdot \nabla_{r_o,\theta_o} T_{1_o}$ is zero at $r_o = 1$ and at $r_o = R_{out}$ this equation reduces to

$$Ra_1 \int_0^H \int_0^{2\pi} \int_1^{R_{out}} \left(\nabla_{r_o,\theta_o} T_{1_o} \right)^2 r_o dr_o d\theta_o dz = 0 \tag{6.201}$$

or

$$Ra_1 \int_0^H \int_0^{2\pi} \int_1^{R_{out}} \left(\frac{dT_{1_o}}{dr_o} \right)^2 r_o dr_o d\theta_o dz = 0 \tag{6.202}$$

where T_{1_o} is given by Equation (6.150).

Now $\left(dT_{1_o}/dr_o\right)$ cannot be zero hence Ra_1 must be zero. This is the expected result because we know intuitively that it should make no difference if the rod is shifted to the right (*i.e.*, $\varepsilon > 0$), or to the left (*i.e.*, $\varepsilon < 0$). Likewise, we expect all the odd Ra 's to be zero (*i.e.*, $Ra_3, Ra_5, \ldots$). Hence to learn the effect of a displacement we must go to the second order of ε. But there we will need T_{1_1} and $V_{z_{1_1}}$ and we must determine them by setting $Ra_1 = 0$. Based on this, the domain equations at order ε becomes

$$\nabla^2 T_{1_1} - V_{z_{1_1}} \frac{dT_o}{dz} = 0 \tag{6.203}$$

and

$$\nabla^2 V_{z_{1_1}} + Ra_o \nabla^2_{r_o,\theta_o} T_{1_1} = 0 \tag{6.204}$$

where the boundary conditions to be satisfied are given by Equations (6.164) to (6.171), namely

$$T_{1_1}(z=0) = 0, \quad T_{1_1}(z=H) = 0$$

$$V_{z_{1_1}}(z=0) = 0, \quad V_{z_{1_1}}(z=H) = 0$$

$$\frac{dT_{1_1}}{dr_o}(r_o = 1) = -R_1 \frac{d^2 T_{1_o}}{dr_o^2}, \quad \frac{dT_{1_1}}{dr_o}(r_o = R_{out}) = 0$$

$$\frac{dV_{z_{1_1}}}{dr_o}(r_o = 1) = -R_1 \frac{d^2 V_{z_{1_o}}}{dr_o^2} \text{ and } \frac{dV_{z_{1_1}}}{dr_o}(r_o = R_{out}) = 0$$

To solve this problem we observe that R_1 is equal to $\cos\theta_o$ and that $\left(dT_o/dz\right)$ depends only on z. Hence we set

$$T_{1_1} = \hat{T}_{1_1}(z)\cos\theta_o f_1(r_o) \tag{6.205}$$

and

$$V_{z_{1_1}} = \hat{V}_{z_{1_1}}(z)\cos\theta_o f_1(r_o) \tag{6.206}$$

where f_1 satisfy

$$\frac{1}{f_1}\frac{1}{r_o}\frac{d}{dr_o}\left(r_o\frac{df_1}{dr_o}\right) - \frac{1}{r_o^2} = -k^2 \tag{6.207}$$

and where

$$\hat{T}_{1_1}\left.\frac{df_1}{dr_o}\right|_{r_o=1} = -\hat{T}_{1_o}\left.\frac{d^2 f_o}{dr_o^2}\right|_{r_o=1} \tag{6.208}$$

$$\hat{V}_{z_{1_1}}\left.\frac{df_1}{dr_o}\right|_{r_o=1} = -\hat{V}_{z_{1_o}}\left.\frac{d^2 f_o}{dr_o^2}\right|_{r_o=1} \tag{6.209}$$

and

$$\left.\frac{df_1}{dr_o}\right|_{r_o=R_{out}} = 0 \tag{6.210}$$

Then Equations (6.203) and (6.204) reduces to

$$\hat{V}_{z_{1_1}}\frac{dT_o}{dz} = \left[\frac{d^2}{dz^2} - k^2\right]\hat{T}_{1_1} \tag{6.211}$$

and

$$\left[\frac{d^2}{dz^2} - k^2\right]\hat{V}_{z_{1_1}} = Ra_o k^2 \hat{T}_{1_1} \tag{6.212}$$

where

$$\hat{T}_{1_1}(z=0) = 0\,, \quad \hat{T}_{1_1}(z=H) = 0$$

$$\hat{V}_{z_{1_1}}(z=0) = 0 \text{ and } \hat{V}_{z_{1_1}}(z=H) = 0$$

By comparing these two equations (*i.e.*, Equations (6.211) and (6.212)), with those at the on-centered case, namely

$$\hat{V}_{z_{1_o}} \frac{dT_o}{dz} = \left[\frac{d^2}{dz^2} - k^2 \right] \hat{T}_{1_o} \tag{6.213}$$

and

$$\left[\frac{d^2}{dz^2} - k^2 \right] \hat{V}_{z_{1_o}} = Ra_o k^2 \hat{T}_{1_o} \tag{6.214}$$

we can observe that $\hat{T}_{1_1}$ is multiple of $\hat{T}_{1_o}$ and $\hat{V}_{z_{1_1}}$ is multiple of $\hat{V}_{z_{1_o}}$ viz.,

$$\hat{T}_{1_1} = D\hat{T}_{1_o} \tag{6.215}$$

and

$$\hat{V}_{z_{1_1}} = D\hat{V}_{z_{1_o}} \tag{6.216}$$

where D is an arbitrary constant. Hence the boundary condition of Equation (6.207) at $r_o = 1$ will become

$$\left. \frac{df_1}{dr_o} \right|_{r_o=1} = -\frac{1}{D} \left. \frac{d^2 f_o}{dr_o^2} \right|_{r_o=1} \tag{6.217}$$

As before $\hat{V}_{z_{1_1}}$ can be found in a series, which satisfies all the boundary conditions viz.,

$$\hat{V}_{z_{1_1}} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{H} z \tag{6.218}$$

But we might notice that $\hat{V}_{z_{1_o}}$ and $\hat{T}_{1_o}$ do not appear but we know that $\hat{V}_{z_{1_1}}$ is multiple of $\hat{V}_{z_{1_o}}$ and so the constants b_n are multiple of the constants B_n. Hence the two-term approximation to $\hat{V}_{z_{1_1}}$ can be written as

$$\hat{V}_{z_{1_1}} = D\hat{V}_{z_{1_o}}$$

where $\hat{V}_{z_{1_o}}$ is given by Equation (6.145).

To find f_1 we note that f_1

$$\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o\frac{\partial f_1}{\partial r_o}\right)+\left(k^2-\frac{1}{r_o^2}\right)f_1=0 \tag{6.219}$$

whence f_1 can be written as

$$f_1 = c_1 J_1\left(kr_o\right)+c_2 Y_1\left(kr_o\right) \tag{6.220}$$

where k has been established at order zero (*i.e.*, from the on-centered problem). To determine the constants c_1 and c_2 we turn to the boundary conditions, which are

$$\frac{df_1}{dr_o}\left(r_o=1\right)=-\frac{1}{D}\frac{d^2 f_o}{dr_o^2}\left(r_o=1\right) \tag{6.221}$$

and

$$\frac{df_1}{dr_o}\left(r_o=R_{out}\right)=0 \tag{6.222}$$

By applying these two boundary conditions into Equation (6.220) we get

$$c_1\left.\frac{dJ_1\left(kr_o\right)}{dr_o}\right|_{r_o=1}+c_2\left.\frac{dY_1\left(kr_o\right)}{dr_o}\right|_{r_o=1}=-\frac{1}{D}\left.\frac{d^2 f_o}{dr_o^2}\right|_{r_o=1} \tag{6.223}$$

and

$$c_1\left.\frac{dJ_1\left(kr_o\right)}{dr_o}\right|_{r_o=R_{out}}+c_2\left.\frac{dY_1\left(kr_o\right)}{dr_o}\right|_{r_o=R_{out}}=0 \tag{6.224}$$

Then solve these two equations for c_1 and c_2. Doing this we can write

$$f_1=\frac{\omega}{D}\left.\frac{d^2 f_o}{dr_o^2}\right|_{r_o=1} \tag{6.225}$$

where

$$\omega=-\frac{J_1\left(kr_o\right)\left.\dfrac{dY_1\left(kr_o\right)}{dr_o}\right|_{r_o=R_{out}}-\left.\dfrac{dJ_1\left(kr_o\right)}{dr_o}\right|_{r_o=R_{out}}Y_1\left(kr_o\right)}{\left.\dfrac{dJ_1\left(kr_o\right)}{dr_o}\right|_{r_o=1}\left.\dfrac{dY_1\left(kr_o\right)}{dr_o}\right|_{r_o=R_{out}}-\left.\dfrac{dJ_1\left(kr_o\right)}{dr_o}\right|_{r_o=R_{out}}\left.\dfrac{dY_1\left(kr_o\right)}{dr_o}\right|_{r_o=1}} \tag{6.226}$$

Because we know f_1 then by using Equations (6.205), (6.206), (6.215) and (6.216) we can rewrite T_{1_1} and $V_{z_{1_1}}$ as

$$T_{1_1}=\omega\hat{T}_{1_o}\left.\frac{d^2f_o}{dr_o^2}\right|_{r_o=1}\cos\theta_o \tag{6.227}$$

and

$$V_{z_{1_1}}=\omega\hat{V}_{z_{1_o}}\left.\frac{d^2f_o}{dr_o^2}\right|_{r_o=1}\cos\theta_o \tag{6.228}$$

where $\hat{T}_{1_o}$, $\hat{V}_{z_{1_o}}$ and f_o are obtained earlier and they are

$$\hat{T}_{1_o}=\left(\frac{1}{Ra_ok^2}\left[-\frac{\pi^2}{H^2}B_1\sin\frac{\pi}{H}z-\frac{4\pi^2}{H^2}B_2\sin\frac{2\pi}{H}z\right]-k^2B_1\sin\frac{\pi}{H}z+B_2\sin\frac{2\pi}{H}z\right)$$

$$\hat{V}_{z_{1_o}}=\left(B_1\sin\frac{\pi}{H}z+B_2\sin\frac{2\pi}{H}z\right)$$

and

$$f_o=C\left(-J_o\left(kr_o\right)\left.\frac{dY_o\left(kr_o\right)}{dr_o}\right|_{r_o=1}+\left.\frac{dJ_o\left(kr_o\right)}{dr_o}\right|_{r_o=1}Y_o\left(kr_o\right)\right)$$

As we explained in earlier chapters, the homogeneous solution will not play a part in the Ra_2 as it enters the calculation multiplied by $\cos\theta_o$ which when integrated over 2π will eventually vanish. So we drop the homogeneous part from the solution of T_{1_1}

and $V_{z_{1_1}}$. This means that T_{1_1} and $V_{z_{1_1}}$ given by Equations (6.227) and (6.228) are written in terms of the particular solutions.

Now, to find Ra_2 we turn to Equations (6.172) and (6.173), where by setting $Ra_1 = 0$, they become

$$\nabla^2 T_{1_2} - V_{z_{1_2}} \frac{dT_o}{dz} = 0 \tag{6.229}$$

and

$$\nabla^2 V_{z_{1_2}} + Ra_o \nabla^2_{r_o,\theta_o} T_{1_2} + Ra_2 \nabla^2_{r_o,\theta_o} T_{1_o} = 0 \tag{6.230}$$

and where the boundary conditions to be satisfied are given by Equations (6.174) to (6.181), namely

$$T_{1_2}(z=0)=0, \quad T_{1_2}(z=H)=0$$

$$V_{z_{1_2}}(z=0)=0, \quad V_{z_{1_2}}(z=H)=0$$

$$\frac{dT_{1_2}}{dr_o}(r_o = R_{out}) = 0, \quad \frac{dV_{z_{1_2}}}{dr_o}(r_o = R_{out}) = 0$$

$$\frac{\partial T_{1_2}}{\partial r_o}(r_o = 1) = -2R_1 \frac{\partial^2 T_{1_1}}{\partial r_o^2} - R_1^2 \frac{\partial^3 T_{1_o}}{\partial r_o^3} - R_2 \frac{\partial^2 T_{1_o}}{\partial r_o^2} - 2\frac{\sin\theta_o}{R_o^2}\frac{\partial T_{1_1}}{\partial \theta_o}$$

and

$$\frac{\partial V_{z_{1_2}}}{\partial r_o}(r_o = 1) = -2R_1 \frac{\partial^2 V_{z_{1_1}}}{\partial r_o^2} - R_1^2 \frac{\partial^3 V_{z_{1_o}}}{\partial r_o^3} - R_2 \frac{\partial^2 V_{z_{1_o}}}{\partial r_o^2} - 2\frac{\sin\theta_o}{R_o^2}\frac{\partial V_{z_{1_1}}}{\partial \theta_o}$$

To begin, let us multiply Equation (6.229) by $V_{z_{1_o}}$ and Equation (6.125), at steady state, by $V_{z_{1_2}}$ to get

$$V_{z_{1_o}} \nabla^2 T_{1_2} - V_{z_{1_o}} V_{z_{1_2}} \frac{dT_o}{dz} = 0 \tag{6.231}$$

and

$$V_{z_{1_2}}\nabla^2 T_{1_o} - V_{z_{1_2}} V_{z_{1_o}} \frac{dT_o}{dz} = 0 \tag{6.232}$$

Then, subtract Equation (6.232) from Equation (6.231), to obtain

$$V_{z_{1_o}}\nabla^2 T_{1_2} - V_{z_{1_2}}\nabla^2 T_{1_o} = 0 \tag{6.233}$$

and integrate the result over the volume of the reference domain to get,

$$\int_0^H \int_0^{2\pi} \int_1^{R_{out}} \left(V_{z_{1_o}}\nabla^2 T_{1_2} - V_{z_{1_2}}\nabla^2 T_{1_o} \right) r_o dr_o d\theta_o dz = 0 \tag{6.234}$$

This equation can be written as

$$\int_0^H \int_0^{2\pi} \int_1^{R_{out}} \left(\nabla\cdot\left(V_{z_{1_o}}\nabla T_{1_2}\right) - \nabla V_{z_{1_o}}\cdot\nabla T_{1_2} - \nabla\cdot\left(V_{z_{1_2}}\nabla T_{1_o}\right) + \nabla V_{z_{1_2}}\cdot\nabla T_{1_o} \right) r_o dr_o d\theta_o dz = 0 \tag{6.235}$$

Now use the divergence theorem to obtain

$$\begin{aligned} &\int_0^H \int_0^{2\pi} \left(\vec{n}\cdot\left(V_{z_{1_o}}\nabla T_{1_2}\right)\Big|_1^{R_{out}} - \int_1^{R_{out}} \nabla V_{z_{1_o}}\cdot\nabla T_{1_2} r_o dr_o \right) d\theta_o dz \\ &\qquad - \int_0^H \int_0^{2\pi} \left(\vec{n}\cdot\left(V_{z_{1_2}}\nabla T_{1_o}\right)\Big|_1^{R_{out}} - \int_1^{R_{out}} \nabla V_{z_{1_2}}\cdot\nabla T_{1_o} r_o dr_o \right) d\theta_o dz = 0 \end{aligned} \tag{6.236}$$

Then, because $V_{z_{1_o}}$ and $V_{z_{1_2}}$ are zero at the top and the bottom and due to $\vec{n}\cdot\nabla T_{1_2}$ being zero at $r_o = R_{out}$ and $\vec{n}\cdot\nabla T_{1_o}$ being zero at $r_o = 1$ and at $r_o = R_{out}$ the foregoing equation reduces to

$$\int_0^H \int_0^{2\pi} \vec{n}\cdot\left(V_{z_{1_o}}\nabla T_{1_2}\right)\Big|_{r_o=1} d\theta_o dz = -\int_0^H \int_0^{2\pi} \int_1^{R_{out}} \left(\nabla V_{z_{1_o}}\cdot\nabla T_{1_2} - \nabla V_{z_{1_2}}\cdot\nabla T_{1_o} \right) r_o dr_o d\theta_o dz \tag{6.237}$$

This equation will be useful later.

Now, multiply Equation (6.230) by T_{1_o} and Equation (6.126) by T_{1_2}, to get

$$T_{1_o}\nabla^2 V_{z_{1_2}} + Ra_o T_{1_o}\nabla^2_{r_o,\theta_o} T_{1_2} + Ra_2 T_{1_o}\nabla^2_{r_o,\theta_o} T_{1_o} = 0 \tag{6.238}$$

and

$$T_{1_2}\nabla^2 V_{z_{1_o}} + Ra_o T_{1_2}\nabla^2_{r_o,\theta_o} T_{1_o} = 0 \tag{6.239}$$

Subtract Equation (6.239) from Equation (6.238) and integrate the result over the volume of the reference domain to get,

$$\begin{aligned}\int_0^H\int_0^{2\pi}\int_1^{R_{out}} \Big(& T_{1_o}\nabla^2 V_{z_{1_2}} - T_{1_2}\nabla^2 V_{z_{1_o}} \\ & + Ra_o\left(T_{1_o}\nabla^2_{r_o,\theta_o} T_{1_2} - T_{1_2}\nabla^2_{r_o,\theta_o} T_{1_o}\right) \\ & + Ra_2 T_{1_o}\nabla^2_{r_o,\theta_o} T_{1_o}\Big) r_o dr_o d\theta_o dz = 0\end{aligned} \tag{6.240}$$

By using the divergence theorem we get

$$\begin{aligned}\int_0^H\int_0^{2\pi}\Bigg(& \vec{n}\cdot\left(T_{1_o}\nabla V_{z_{1_2}}\right)\Big|_1^{R_{out}} - \int_1^{R_{out}} \nabla T_{1_o}\cdot\nabla V_{z_{1_2}} r_o dr_o - \vec{n}\cdot\left(T_{1_2}\nabla V_{z_{1_o}}\right)\Big|_1^{R_{out}} + \int_1^{R_{out}} \nabla T_{1_2}\cdot\nabla V_{z_{1_o}} r_o dr_o \\ & + Ra_o\Bigg(\vec{n}\cdot\left(T_{1_o}\nabla_{r_o,\theta_o} T_{1_2}\right)\Big|_1^{R_{out}} - \int_1^{R_{out}} \nabla_{r_o,\theta_o} T_{1_o}\cdot\nabla_{r_o,\theta_o} T_{1_2} r_o dr_o \\ & \qquad - \vec{n}\cdot\left(T_{1_2}\nabla_{r_o,\theta_o} T_{1_o}\right)\Big|_1^{R_{out}} + \int_1^{R_{out}} \nabla_{r_o,\theta_o} T_{1_2}\cdot\nabla_{r_o,\theta_o} T_{1_o} r_o dr_o\Bigg) \\ & + Ra_2\Bigg(\vec{n}\cdot\left(T_{1_o}\nabla_{r_o,\theta_o} T_{1_o}\right)\Big|_1^{R_{out}} - \int_1^{R_{out}} \nabla_{r_o,\theta_o} T_{1_o}\cdot\nabla_{r_o,\theta_o} T_{1_o} r_o dr_o\Bigg)\Bigg) d\theta_o dz = 0\end{aligned} \tag{6.241}$$

Now, because T_{1_o} and T_{1_2} are zero at the top and the bottom and due to the fact that $\vec{n}\cdot\nabla T_{1_o}$ and $\vec{n}\cdot\nabla V_{z_{1_o}}$ are both zero at $r_o = 1$ and at $r_o = R_{out}$ and because $\vec{n}\cdot\nabla V_{z_{1_2}}$ is zero at $r_o = R_{out}$ and by using Equation (6.237) viz.,

$$\int_0^H\int_0^{2\pi}\vec{n}\cdot\left(V_{z_{1_o}}\nabla T_{1_2}\right)\Big|_{r_o=1} d\theta_o dz = -\int_0^H\int_0^{2\pi}\int_1^{R_{out}}\left(\nabla V_{z_{1_o}}\cdot\nabla T_{1_2} - \nabla V_{z_{1_2}}\cdot\nabla T_{1_o}\right) r_o dr_o d\theta_o dz$$

we can rewrite Equation (6.241) as

$$Ra_2 = -\frac{\int_0^H \int_0^{2\pi} \left(\vec{n} \cdot \left(T_{1_o} \nabla V_{z_{1_2}} \right)\Big|_{r_o=1} + \vec{n} \cdot \left(V_{z_{1_o}} \nabla T_{1_2} \right)\Big|_{r_o=1} + Ra_o \vec{n} \cdot \left(T_{1_o} \nabla_{r_o,\theta_o} T_{1_2}\Big|_{r_o=1} \right) \right) d\theta_o dz}{\int_0^H \int_0^{2\pi} \int_1^{R_{out}} \left(\nabla_{r_o,\theta_o} T_{1_o} \right)^2 r_o dr_o d\theta_o dz} \tag{6.242}$$

or

$$Ra_2 = -\frac{\int_0^H \int_0^{2\pi} \left(T_{1_o} \frac{dV_{z_{1_2}}}{dr_o}\Big|_{r_o=1} + V_{z_{1_o}} \frac{dT_{1_2}}{dr_o}\Big|_{r_o=1} + Ra_o T_{1_o} \frac{dT_{1_2}}{dr_o}\Big|_{r_o=1} \right) d\theta_o dz}{\int_0^H \int_0^{2\pi} \int_1^{R_{out}} \left(\frac{dT_{1_o}}{dr_o} \right)^2 r_o dr_o d\theta_o dz} \tag{6.243}$$

The denominator is always positive but we cannot decide the sign of the nominator without a calculation. This calculation is made at different values of R_{out} and H. Some results are illustrated in Table 6-2.

Table 6-2. Values of Ra_o and Ra_2 at different values of R_{out} and H.

R_{out}	H	Ra_o	Ra_2
1.5	5	46.57	-3.93×10^4
	10	22.46	-1.90×10^4
	15	14.87	-1.30×10^4
5.0	5	3.950	-1.110
	10	0.703	-0.200
	15	0.351	-0.098

These results suggest that the value of $Ra_{critical}$ decreases as we shift the rod off-center and hence it is easier to start natural convection when the rod is displaced. It is important to note that Ra (experimental) depends on R_{in} (Ra increases as R_{in} increases), while $Ra_{critical}$ does not depend on R_{in} and, hence, the bigger the region make it easier for natural convection to occur. From the results presented in Table 6-2, we can observe that at a given value of R_{out} the values of Ra_o decreases as the annular region become deeper (*i.e.*, H increases), and at a given value of H the value of Ra_o decreases as the annular region become wider (*i.e.*, R_{out} increases). These results are expected because the deeper the annular region is the harder it becomes for the heat to get out through the top and the bottom walls and this will increase the temperature and if the temperature rises enough to make the gradient stronger then this will increase the density gradient and therefore it become easier for natural convection to take place. On the other hand, a wider annular region can accommodate more flow cells than a thin region and this will lower $Ra_{critical}$. Notice that even with setting Q equal to a constant the calculations are not ordinarily simple.

Now, let us summarize what we have done and learned in this chapter. The aim of this chapter was to steady the thermal stability in porous media with natural convection. To achieve this, we begin by considering a porous material that produces heat in an annular region where the heat source depends linearly on temperature. On steadying this problem we learned that, in the on-center case, an adverse temperature gradient could induce a natural convection that may enhance the thermal stability of the system and for this to happen the problem needs to exceed a certain value of Rayleigh number (*i.e.*, $Ra_{critical}$), that is defined in terms of the parameters of the problem itself. Then, as we shift the rod off-center, we found that it becomes difficult to solve this problem and the usual method (*i.e.*, perturbing a simpler eigenvalue problem), that we used in the earlier chapters do not work here because this problem does not have a simple resolution to it. However, to learn more about thermal stability of this kind of problems we carried out the calculation but with approximation. This approximation is to hold the heat source fixed. By solving this modified problem we found that, when the inner cylinder is displaced slightly from the center of the annulus the chance for natural convection to come into action increases. Unfortunately, in these problems we were not able to show that thermal stability enhances as natural convection begins to take place within the region. This is because to show that we must calculate the total heat-transfer rate in the case where natural convection can occur and in the case where no natural convection can take place to see the difference. But we could not do that because we do not know the exact value of the perturbed temperature except through an arbitrary constant. In addition to that we learned that even for this simple problem the calculations that determine the critical value of the Rayleigh number are not ordinarily simple.

CHAPTER 7
CONCLUSIONS

The main aim of this dissertation was to study the effect of geometry change on the thermal stability of heat-producing regions where the heat source depends linearly on temperature. We defined the thermal instability (*i.e.*, thermal ignition or explosion), by the occurrence of an infinite maximum temperature, which in turn translates into our inability to find a steady solution. In this concluding chapter, we will present the most important of the results that we have obtained from this study.

To understand how the geometry change can affect the thermal stability of heat producing regions we used the domain and surface perturbation methods. Using such methods we concluded that a distorted annulus could tolerate more heat generation than a perfectly centered one. In solving a problem on the effect of geometry change on the thermal stability of a planar geometry with heat being produced within the domain we found that when the unboundedness or non-existence is used to identify the instability the domain perturbation method cannot be applied directly to the domain equations. Instead we must introduce the corresponding eigenvalue problem and solve it in terms of the reference geometry. In other wards, the eigenvalue problem is solved by perturbing the domain equations about the reference state. On doing this, we found that the thermal instability is given by the condition when the leading eigenvalue becomes zero. In addition to this, we found that the value of the diffusion strength becomes an indicator of the thermal stability, that is: as the diffusion strength increases the thermal stability of the system increases. We also learned that thin heat-producing regions are thermally more stable than thick heat-producing regions.

Then series of problems were considered. There was the thermal stability of a heat-generating annular region under a variety of boundary conditions without flow and with flow. By solving this problem we found that for the cases of constant inner wall temperature and at adiabatic inner wall, the annular region becomes thermally less stable as the inner rod is shifted from the center of the geometry, but for the case of perfectly conducting inner wall the opposite is true (*i.e.*, the distorted annular region is thermally more stable than a symmetric one). We also learned that, regardless of the reduction in its thermal stability due to the displacement, the region with constant inner wall temperature is always more stable than the other two cases. Also we found that the region with adiabatic inner wall is slightly less stable than the region with perfectly conducting inner wall. Furthermore, we found that the thermal stability of the region with perfectly conducting inner wall is weakly dependent on the location of the inner cylinder. From all of this we learned that, the thermal stability of a heat-generating annular region depends strongly on the location of the inner cylinder and on the boundary conditions of the region. Thus, in designing heat exchangers, which are typically designed in a concentric cylinder form, the designer must be very careful to get the alignment right and to take into consideration the boundary conditions of the system.

The heat transfer assumption was then modified as a little forced convection is added. To study this case we added a radial flow to the case of constant walls temperatures and then assumed that the fluid flow is described via Darcy's law. Doing this we found that, at the reference state, the radial flow, regardless of its direction, increases the thermal stability of this problem. But on perturbing the inner cylinder the thermal stability decreases and it becomes a function of the direction of the radial flow, that is: the case with outward flow is thermally more stable than the case with inward flow. Also, we found that at high flow the correction factors cannot be important and the problem always remains stable. In addition to this we were able to show by using simple derivations that this problem must always be stable regardless of the direction of the flow or its strength. From this we convinced our selves that the thermal stability of this problem strongly depends on the heat-transfer mode.

Viscous flow problems were considered next to show how the results presented earlier tie into some concrete physical problems. To achieve this, we considered viscous heating of liquids in simple flows and we started by showing that viscous fluid flow problems are in fact thermal ignition problems in which the heat source depends linearly on temperature. Some auxiliary problems were studied and solved to get some idea of the effect of unequal temperatures at the boundaries and of the effect of geometry change in frictional heating. From these auxiliary problems we learned that, the thermal stability of viscous heating does not depend on whether or not the two boundaries are at different temperatures even though the details of the flow and the temperature will be affected by the temperature difference at the boundaries. In addition to this we learned that the thermal stability of viscous heating problems is strongly dependent on their geometry. In addition to all this another interesting problem was considered. This problem is the thermal stability of frictional heating in axial flow between a rod and a pipe due to viscous heating. On studying this problem we found that, this problem can be solved only when it is perfectly symmetric but as the inner rod is shifted slightly from the center of the annulus the mathematics of the problem become very complicated. However, to learn more about thermal stability of this geometry we carried out the calculation but with the approximation that the viscosity was fixed. By solving this modified problem we found that this problem is always thermally stable and when the inner cylinder is displaced slightly from the center of the annulus we found that both the total volumetric flow rate and the total heat-transfer rate increase. The increases in these two rates have two opposite effects on the thermal stability of this problem (*i.e.*, the increase in total volumetric flow rate reduces the thermal stability of the problem while the increase in the total heat-transfer rate enhances its thermal stability). It would appear that, because the problem is always stable, either the two effects just offset leaving the thermal stability of the problem unchanged or the stabilizing effect will be the predominant effect. Also we found that the location of the maximum temperature strongly depends on the shape of the geometry.

To give another physical application to the results presented earlier additional problem was considered. This problem is to study thermal stability in a porous media with natural convection. On studying this problem we learned that, in the on-center case, an adverse temperature gradient could induce natural convection that can then enhance the thermal stability of the system and for this to happen the problem needs to exceed a certain value of Rayleigh number. Then, as we shift the rod off-center, we found that it becomes difficult to solve this problem and the usual method (*i.e.*, perturbing a simpler eigenvalue problem), did not work. Again, to learn more about thermal stability of these kinds of problems we carried out the calculation with an approximation, that is: to hold the heat source fixed. By solving this modified problem we found that, when the inner cylinder is displaced slightly from the center of the annulus the chance for natural convection to come into action increases. Unfortunately, in these problems we were not able to prove that thermal stability enhances as natural convection begins to take place within the region. This was because the value of the perturbed temperature is known only up to an arbitrary constant.

In conclusion, in this dissertation, the physics of thermal stability under different imposed restrictions and various settings has been explored with an idea of understanding the physics as the geometry is changed or distorted.

APPENDIX A

EXPLINATION OF THE DOMAIN PERTURBATION METHOD REQUIRED TO CARRY OUT THE CALCULATION OF THIS WORK

This appendix gives an explanation for the domain perturbation method that is used to solve the main problems considered in this thesis (*i.e.*, problems with distorted geometry). This explanation is based on the work of Morse, P.M. and Feshbach, H. (1953) and on our understanding of the work of Johns, L.E. and Narayanan, R. (2002). To begin, let u denote the solution to a problem defined on a domain D, where equations sufficient in number are specified on D, and on its boundary, to determine u. Among problems like this are those where the domain D itself presents a major challenge. These problems are of two kinds. Those where D is not specified and must be determined as part of the solution, and those where D is specified but is irregular.[1] The number of equations to be satisfied can be different in the two cases.

In any of these two cases, estimates of u, or of u and D, can be obtained by solving a simple problem defined on a specified and regular domain. The domain of this simple problem, denoted by D_o, is called the reference domain. It may be determined by the original problem or it may be set so as to simplify our work.

Our job is to find out how to modify the problem on D to make up for the change in its domain to D_o. Particularly, it is to determine u in terms of the solutions to problems defined on the reference domain D_o. The first part of our job is to decide what problems on D_o might help us.

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[1] By the word irregular we do not mean to imply not smooth, but instead, not convenient.

To explain how to do this and to show exactly how all of the details of the calculations work out, let the problem be dimensional and let us work in terms of Cartesian coordinates. Then, in terms of an original O and axes Ox and Oy at right angles, the points of D_o will be denoted by coordinates x_o, y_o and those of D by coordinates x, y.

Now, the main idea is to imagine a family of domains D_ε growing out of the reference domain D_o and to imagine that u must be determined on each of these, one being the domain of interest. The points (x, y) of the domain D_ε are then determined in terms of the points (x_o, y_o) of the reference domain D_o by the *mapping*

$$x = f(x_o, y_o, \varepsilon) \tag{A.1}$$

and

$$y = g(x_o, y_o, \varepsilon) \tag{A.2}$$

Based on this, each point of D_ε gets an ancestor in D_o and determining the mapping of D_o into D_ε may be part of solving the problem.

Now, let D_ε be unspecified or irregular in only one part of its boundary and let D_o be specified and regular at this boundary. Then, let the domains D_ε grow out of the reference domain D_o as part of the boundary of D_o, denoted $y_o = Y_o(x_o)$, passes through a series of configurations $y = Y(x, \varepsilon)$ forming the corresponding parts of the boundaries of the domains D_ε. It is sufficient to simplify the mapping of D_o into D_ε to

$$x = x_o$$

and

$$y = g\left(x_o, y_o, \varepsilon\right)$$

where this mapping must carry out the points $\left(x_o, y_o = Y_o\left(x_o\right)\right)$ of the boundary of the reference domain D_o into the corresponding points $\left(x, y = Y\left(x, \varepsilon\right)\right)$ of the boundary of D_ε, namely

$$x = x_o$$

and

$$y = Y\left(x, \varepsilon\right) = Y\left(x_o, \varepsilon\right) = g\left(x_o, Y_o\left(x_o\right), \varepsilon\right)$$

This simple mapping is the foundation of our work (*i.e.*, to perform the perturbation calculations when solving an unknown domain), and it is depicted in Figure A-1.

The strategy of this appendix is to work out the expansion along the mapping of a variable u on the domain D_ε in terms of variables $u_o, u_1, u_2, \ldots$ on the reference domain D_o.

To put this strategy to work, the function g is expanded in power of ε as

$$g\left(x_o, y_o, \varepsilon\right) = g\left(x_o, y_o, \varepsilon = 0\right) + \varepsilon \frac{\partial g}{\partial \varepsilon}\left(x_o, y_o, \varepsilon = 0\right) + \frac{1}{2}\varepsilon^2 \frac{\partial^2 g}{\partial \varepsilon^2}\left(x_o, y_o, \varepsilon = 0\right) + \ldots$$

where $g\left(x_o, y_o, \varepsilon = 0\right) = y_o$ and where the derivatives of g, evaluated in the reference domain, are taken holding x_o and y_o fixed. Then, in terms of notation

$$y_1\left(x_o, y_o\right) = \frac{\partial g}{\partial \varepsilon}\left(x_o, y_o, \varepsilon = 0\right) \tag{A.3}$$

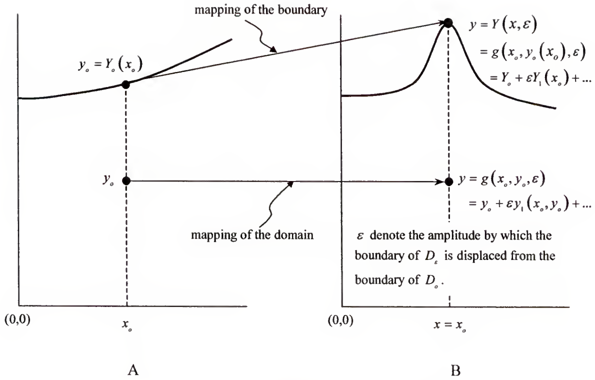
mapping of the boundary
$y_o = Y_o(x_o)$
$y = Y(x,\varepsilon)$
$= g(x_o, y_o(x_o), \varepsilon)$
$= Y_o + \varepsilon Y_1(x_o) + \ldots$
y_o
$y = g(x_o, y_o, \varepsilon)$
$= y_o + \varepsilon y_1(x_o, y_o) + \ldots$
mapping of the domain
ε denote the amplitude by which the boundary of D_ε is displaced from the boundary of D_o.
(0,0)
x_o
(0,0)
$x = x_o$
A
B

Figure A-1. Mapping D_o into D_ε. A) Reference domain D_o. B) Present domain D_ε.

$$y_2(x_o, y_o) = \frac{\partial^2 g}{\partial \varepsilon^2}(x_o, y_o, \varepsilon = 0) \tag{A.4}$$

$$\vdots$$

etc.

then the mapping can be written as

$$x = x_o \tag{A.5}$$

and

$$y = g(x_o, y_o, \varepsilon) = y_o + \varepsilon y_1(x_o, y_o) + \frac{1}{2}\varepsilon^2 y_2(x_o, y_o) + \ldots \tag{A.6}$$

and this is the form in which it is used.

Now, the boundary of the reference domain must be carried into the boundary of the present domain by our mapping. Hence, the function $Y(x,\varepsilon)$, which describes the boundary of the current domain, inherits its expansion in powers of ε by putting $y_o = Y_o(x_o)$ into the expansion of the domain mapping. Then, at the boundary, the mapping

$$x = x_o$$

and

$$y = Y(x,\varepsilon) = Y_o(x_o) + \varepsilon y_1(x_o, Y_o(x_o)) + \frac{1}{2}\varepsilon^2 y_2(x_o, Y_o(x_o)) + \ldots$$

which can be written as

$$x = x_o \tag{A.7}$$

and

$$y = Y(x,\varepsilon) = Y_o(x_o) + \varepsilon Y_1(x_o) + \frac{1}{2}\varepsilon^2 Y_2(x_o) + \ldots \tag{A.8}$$

by introducing the notation

$$Y_1(x_o) = y_1(x_o, Y_o(x_o)) \tag{A.9}$$

$$Y_2(x_o) = y_2(x_o, Y_o(x_o)) \tag{A.10}$$

$$\vdots$$

etc.

It is $Y_1, Y_2, \ldots$ that must be determined or assigned to specify the domain D_ε in terms of the reference domain D_o.

Now, to determine the expansion of a domain variable and its derivatives along the mapping, note that there are two kinds of variables of interest to us: those defined on the domain, and hence on its boundary as well, and those defined only on the boundary of the domain. These latter variables ordinarily depend on the shape of the boundary and acquire their expansion by the expansion of the function determining this shape. The curvature of the bounding surface, the normal and the normal speed are examples of variables defined only on the boundary, but many variables of interest to us are defined on the domain as well as on its boundary and it is to these variable that we turn first.

Let u, as it depends on x, y and ε, denote the solution to a family of problems on a family of domains, denoted D_ε. The domains D_ε grow out of a reference domain D_o. A point (x, y) of the domain D_ε is connected to its ancestor (x_o, y_o) of D_o by Equations (A.5) and (A.6). Let $u(x_o, y_o, \varepsilon = 0)$ be denoted by $u_o(x_o, y_o)$. It is understood that the problem satisfied by u can be solved on D_o to produce u_o and D_o too if it is not known. Our plan is to determine u at the point (x, y) in D_ε in terms of u_o, and whatever else is required, at the point (x_o, y_o) in D_o.

To find out what is required, we expand $u(x, y, \varepsilon)$ in powers of ε along the mapping and write this expansion

$$\begin{aligned} u(x,y,\varepsilon) &= u(x = x_o, y = y_o, \varepsilon = 0) + \varepsilon \frac{du}{d\varepsilon}(x = x_o, y = y_o, \varepsilon = 0) \\ &\quad + \frac{1}{2}\varepsilon^2 \frac{d^2u}{d\varepsilon^2}(x = x_o, y = y_o, \varepsilon = 0) + \ldots \end{aligned} \tag{A.11}$$

where $d/d\varepsilon$ denotes the derivative of a function depending on x, y and ε taken along the mapping (*i.e.*, taken holding x_o and y_o fixed where x and y depend on x_o, y_o and ε by the use of the mapping), viz., via Equations (A.5) and (A.6).

Then, with u_o defined by

$$u_o(x_o, y_o) = u(x = x_o, y = y_o, \varepsilon = 0)$$

work out $\dfrac{du}{d\varepsilon}$ and $\dfrac{d^2u}{d\varepsilon^2}$.

To obtain a formula for $\dfrac{du}{d\varepsilon}(x = x_o, y = y_o, \varepsilon = 0)$, differentiate u along the mapping taking x and y to depend on ε, holding x_o and y_o fixed. By the chain rule, there results

$$\frac{du}{d\varepsilon}(x,y,\varepsilon) = \frac{\partial u}{\partial \varepsilon}(x,y,\varepsilon) + \frac{\partial u}{\partial y}(x,y,\varepsilon)\frac{\partial g}{\partial \varepsilon}(x_o,y_o,\varepsilon) \tag{A.12}$$

where $u = g(x_o, y_o, \varepsilon)$ and where the term $\dfrac{\partial u}{\partial x}(x,y,\varepsilon)\dfrac{\partial f}{\partial \varepsilon}(x_o,y_o,\varepsilon)$, that might have been expected had Equation (A.1) been used, does not appear due to the fact that Equation (A.5), namely

$$x = f(x_o, y_o, \varepsilon) = x_o$$

is being used instead, where x_o is being held fixed. Then, set ε to zero in Equation (A.12) and use Equation (A.3) to obtain

$$\frac{du}{d\varepsilon}(x = x_o, y = y_o, \varepsilon = 0) = \frac{\partial u}{\partial \varepsilon}(x_o, y_o, \varepsilon = 0) + y_1(x_o, y_o)\frac{\partial u}{\partial y}(x_o, y_o, \varepsilon = 0) \tag{A.13}$$

In Equation (A.13), introduce u_1 via

$$u_1(x_o, y_o) = \frac{\partial u}{\partial \varepsilon}(x = x_o, y = y_o, \varepsilon = 0)$$

and observe that

$$\frac{\partial u}{\partial y}(x = x_o, y = y_o, \varepsilon = 0) = \frac{\partial u_o}{\partial y_o}(x_o, y_o)$$

to get

$$\frac{du}{d\varepsilon}\left(x=x_o,y=y_o,\varepsilon=0\right)=u_1\left(x_o,y_o\right)+y_1\left(x_o,y_o\right)\frac{\partial u_o}{\partial y_o}\left(x_o,y_o\right) \qquad \text{(A.14)}$$

where y_1 specifies the mapping to first order of ε and where u_1, like u_o, remains to be determined.

To work out the second derivative, differentiate Equation (A.12) to obtain

$$\frac{d}{d\varepsilon}\left[\frac{du}{d\varepsilon}\right]=\frac{d}{d\varepsilon}\left[\frac{\partial u}{\partial\varepsilon}+\frac{\partial u}{\partial y}\frac{\partial g}{\partial\varepsilon}\right]=\frac{d}{d\varepsilon}\left[\frac{\partial u}{\partial\varepsilon}\right]+\frac{d}{d\varepsilon}\left[\frac{\partial u}{\partial y}\right]\frac{\partial g}{\partial\varepsilon}+\frac{\partial u}{\partial y}\frac{\partial^2 g}{\partial\varepsilon^2}$$

and substitute

$$\frac{d}{d\varepsilon}\left[\frac{\partial u}{\partial\varepsilon}\right]=\frac{\partial}{\partial\varepsilon}\left[\frac{\partial u}{\partial\varepsilon}\right]+\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial\varepsilon}\right]\frac{\partial g}{\partial\varepsilon}$$

and

$$\frac{d}{d\varepsilon}\left[\frac{\partial u}{\partial y}\right]=\frac{\partial}{\partial\varepsilon}\left[\frac{\partial u}{\partial y}\right]+\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\right]\frac{\partial g}{\partial\varepsilon}$$

to obtain

$$\frac{d^2u}{d\varepsilon^2}=\frac{\partial^2u}{\partial\varepsilon^2}+2\frac{\partial^2u}{\partial y\partial\varepsilon}\frac{\partial g}{\partial\varepsilon}+\frac{\partial^2u}{\partial y^2}\left[\frac{\partial g}{\partial\varepsilon}\right]^2+\frac{\partial u}{\partial y}\frac{\partial^2 g}{\partial\varepsilon^2} \qquad \text{(A.15)}$$

Then, in Equation (A.15), set $x=x_o$, $y=y_o$ and $\varepsilon=0$, introduce u_2 via

$$u_2\left(x_o,y_o\right)=\frac{\partial^2u}{\partial\varepsilon^2}\left(x=x_o,y=y_o,\varepsilon=0\right)$$

observe that

$$\frac{\partial^2u}{\partial y^2}\left(x=x_o,y=y_o,\varepsilon=0\right)=\frac{\partial^2u_o}{\partial y_o^2}\left(x_o,y_o\right)$$

and use Equations (A.3) and (A.4) to obtain

$$\frac{d^2u}{d\varepsilon^2}\left(x=x_o,y=y_o,\varepsilon=0\right)=u_2\left(x_o,y_o\right)+2y_1\left(x_o,y_o\right)\frac{\partial u_1}{\partial y_o}\left(x_o,y_o\right)$$
$$+y_1^2\left(x_o,y_o\right)\frac{\partial^2 u_o}{\partial y_o^2}\left(x_o,y_o\right)+y_2\left(x_o,y_o\right)\frac{\partial u_o}{\partial y_o}\left(x_o,y_o\right) \qquad \text{(A.16)}$$

where y_2 specifies the mapping to second order of ε and where u_2 remains to be determined.

Additional derivatives needed to write $u(x,y,\varepsilon)$ to any order in ε can be determined by carrying this work further; indeed, their forms can even be guessed, but Equations (A.14) and (A.16) are enough to write u to second order in ε as

$$u(x,y,\varepsilon)=u_o+\varepsilon\left[u_1+y_1\frac{\partial u_o}{\partial y_o}\right]+\frac{1}{2}\varepsilon^2\left[u_2+2y_1\frac{\partial u_1}{\partial y_o}+y_1^2\frac{\partial^2 u_o}{\partial y_o^2}+y_2\frac{\partial u_o}{\partial y_o}\right]+\dots \qquad \text{(A.17)}$$

where every variable on the right-hand side is evaluated at (x_o,y_o).

To determine u on the domain D_ε, then, it would seem as though $u_o,u_1,u_2,\dots$ as well as $y_1,y_2,\dots$ would need to be determined on the reference domain D_o. It turns out that while $u_o,u_1,u_2,\dots$ can be determined, $y_1,y_2,\dots$ cannot, and, although this presents a technical difficulty, it is helpful to be guided by the idea that what is required is a mapping-independent prediction of the values of u at the interior points of the domain D_ε.

One aim, then, is to derive the equations that must be satisfied by the functions $u_o,u_1,u_2,\dots$ on the reference domain D_o from the equations satisfied by u on the present domain D_ε. Now, derivatives of u appear in most of the equations of interest; hence, in addition to the expansion of u, expansions of its derivatives *e.g.*, expansions of $\partial u/\partial x$

and $\partial u/\partial y$, must be obtained. These can be used along with the expansion of u in determining the equations satisfied by $u_o, u_1, u_2, \ldots$ on the reference domain D_o.

Taking $\partial u/\partial y$ first, we must differentiate the right-hand side of Equation (A.17) with respect to y, holding x and ε fixed, where the functions appearing there depend on x_o, y_o and ε. To do this, Equations (A.5) and (A.6) can be used to obtain x_o and y_o in terms of x, y and ε, and then the chain rule of differentiation can be used. In this use of the chain rule, functions of x_o, y_o and ε are to be differentiated with respect to y, holding x and ε fixed; earlier, to obtain Equations (A.14) and (A.16), functions of x, y and ε were differentiated with respect to ε, holding x_o and y_o fixed.

The chain rule, then, requires functions of x_o and y_o to be differentiated according to

$$\frac{\partial}{\partial y}[\ ] = \frac{\partial}{\partial x_o}[\ ]\frac{\partial x_o}{\partial y} + \frac{\partial}{\partial y_o}[\ ]\frac{\partial y_o}{\partial y}$$

but $\partial x_o/\partial y$ is zero, due to the fact that $x = f(x_o, y_o, \varepsilon) = x_o$, and by this, the chain rule reduces to

$$\frac{\partial}{\partial y}[\ ] = \frac{\partial}{\partial y_o}[\ ]\frac{\partial y_o}{\partial y}$$

Hence, we must differentiate the right-hand side of Equation (A.17) with respect to y_o, holding x_o and ε fixed, and then multiply this by $\partial y_o/\partial y$ which comes from Equation (A.6), holding x and ε fixed. Doing this leads to

$$\begin{aligned}\frac{\partial u}{\partial y}=&\left[\frac{\partial u_o}{\partial y_o}\frac{\partial y_o}{\partial y}\right]+\varepsilon\left[\frac{\partial u_1}{\partial y_o}\frac{\partial y_o}{\partial y}+y_1\frac{\partial^2 u_o}{\partial y_o^2}\frac{\partial y_o}{\partial y}+\frac{\partial y_1}{\partial y}\frac{\partial u_o}{\partial y_o}\right]\\&+\frac{1}{2}\varepsilon^2\left[\frac{\partial u_2}{\partial y_o}\frac{\partial y_o}{\partial y}+2y_1\frac{\partial^2 u_1}{\partial y_o^2}\frac{\partial y_o}{\partial y}+2\frac{\partial y_1}{\partial y}\frac{\partial u_1}{\partial y_o}+y_1^2\frac{\partial^3 u_o}{\partial y_o^3}\frac{\partial y_o}{\partial y}\right.\\&\left.+2y_1\frac{\partial y_1}{\partial y}\frac{\partial^2 u_o}{\partial y_o^2}+y_2\frac{\partial^2 u_o}{\partial y_o^2}\frac{\partial y_o}{\partial y}+\frac{\partial y_2}{\partial y}\frac{\partial u_o}{\partial y_o}\right]+\ldots\end{aligned}\qquad\text{(A.18)}$$

where the chain rule has been used to differentiate $u_o, \partial u_o/\partial y_o, \partial^2 u_o/\partial y_o^2, u_1, \partial u_1/\partial y_o$ and u_2, but it has not been used to differentiate y_1 and y_2. These derivatives have been indicated as $\partial y_1/\partial y$ and $\partial y_2/\partial y$ for reasons to appear presently.

Now, Equations (A.5) and (A.6) can be used to determine $\partial y_o/\partial y$ as

$$\frac{\partial y_o}{\partial y}=1-\varepsilon\frac{\partial y_1}{\partial y}-\frac{1}{2}\varepsilon^2\frac{\partial y_2}{\partial y}-\ldots$$

and this reduces Equation (A.18), the expansion of $\partial u/\partial y$, to its final form, that is

$$\begin{aligned}\frac{\partial u}{\partial y}(x,y,\varepsilon)=&\frac{\partial u_o}{\partial y_o}+\varepsilon\left[\frac{\partial u_1}{\partial y_o}+y_1\frac{\partial^2 u_o}{\partial y_o^2}\right]\\&+\frac{1}{2}\varepsilon^2\left[\frac{\partial u_2}{\partial y_o}+2y_1\frac{\partial^2 u_1}{\partial y_o^2}+y_1^2\frac{\partial^3 u_o}{\partial y_o^3}+y_2\frac{\partial^2 u_o}{\partial y_o^2}\right]+\ldots\end{aligned}\qquad\text{(A.19)}$$

The main thing to observe in this formula is that all the terms in Equation (A.18) in which derivatives of $y_1, y_2, \ldots$ appear *e.g.*, $\frac{\partial y_1}{\partial y}\frac{\partial u_o}{\partial y_o}$, are lost in taking the last step, with the result that the expansion of $\partial u/\partial y$ looks just like the expansion of u, as might have been anticipated, namely

$$u=u_o+\varepsilon\left[u_1+y_1\frac{\partial u_o}{\partial y_o}\right]+\frac{1}{2}\varepsilon^2\left[u_2+2y_1\frac{\partial u_1}{\partial y_o}+y_1^2\frac{\partial^2 u_o}{\partial y_o^2}+y_2\frac{\partial u_o}{\partial y_o}\right]+\ldots$$

and

$$\frac{\partial u}{\partial y}=\frac{\partial u_o}{\partial y_o}+\varepsilon\left[\frac{\partial u_1}{\partial y_o}+y_1\frac{\partial^2 u_o}{\partial y_o^2}\right]+\frac{1}{2}\varepsilon^2\left[\frac{\partial u_2}{\partial y_o}+2y_1\frac{\partial^2 u_1}{\partial y_o^2}+y_1^2\frac{\partial^3 u_o}{\partial y_o^3}+y_2\frac{\partial^2 u_o}{\partial y_o^2}\right]+\ldots$$

and this would lead us to expect the expansion of $\frac{\partial^2 u}{\partial y^2}$ to be

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u_o}{\partial y_o^2}+\varepsilon\left[\frac{\partial^2 u_1}{\partial y_o^2}+y_1\frac{\partial^3 u_o}{\partial y_o^3}\right]&\\+\frac{1}{2}\varepsilon^2\left[\frac{\partial^2 u_2}{\partial y_o^2}+2y_1\frac{\partial^3 u_1}{\partial y_o^3}+y_1^2\frac{\partial^4 u_o}{\partial y_o^4}+y_2\frac{\partial^3 u_o}{\partial y_o^3}\right]+\ldots&\end{aligned} \tag{A.20}$$

which, indeed, is correct. In all these formulas, the variables on the left-hand side are evaluated at x, y and ε whereas all the variables on the right-hand side are evaluated at x_o and y_o.

Turning to $\partial u/\partial x$, we must differentiate the right-hand side of Equation (A.17) with respect x, holding y and ε fixed. Now, x is equal to x_o, but y is being held fixed; hence, y_o cannot be fixed. The result is that $\partial/\partial x$, where y is held fixed, operating on the right-hand side cannot be replaced by $\partial/\partial x_o$, where y_o is being held fixed, even though $x = x_o$. The mapping must be used to determine y_o as a function of x at a fixed values of y and ε. Again, all of the variables on the right-hand side of the expansion of $u(x, y, \varepsilon)$, outside of ε itself, which is being held fixed, depend only on x_o and y_o. In view of this, our calculation of $\partial u/\partial x$ again requires the use of the chain rule, and in this case it has the form

$$\frac{\partial}{\partial x}[\ ]=\frac{\partial}{\partial x_o}[\ ]\frac{\partial x_o}{\partial x}+\frac{\partial}{\partial y_o}[\ ]\frac{\partial y_o}{\partial x}$$

Once more using Equations (A.5) and (A.6) to express x_o and y_o in terms of x and y, there now obtain

$$\frac{\partial x_o}{\partial x} = 1$$

and

$$0 = \frac{\partial y_o}{\partial x} + \varepsilon \frac{\partial y_1}{\partial x} + \frac{1}{2}\varepsilon^2 \frac{\partial y_2}{\partial x} + \ldots$$

whence the chain rule takes the form

$$\frac{\partial}{\partial x}[\;] = \frac{\partial}{\partial x_o}[\;] + \frac{\partial}{\partial y_o}[\;]\frac{\partial y_o}{\partial x}$$

where

$$\frac{\partial y_o}{\partial x} = -\varepsilon \frac{\partial y_1}{\partial x} - \frac{1}{2}\varepsilon^2 \frac{\partial y_2}{\partial x} - \ldots \tag{A.21}$$

Working out $\partial/\partial x$ applied to the right-hand side of Equation (A.17) then leads to

$$\begin{aligned}
\frac{\partial u}{\partial x} = & \left[\frac{\partial u_o}{\partial x_o} + \frac{\partial u_o}{\partial y_o}\frac{\partial y_o}{\partial x}\right] + \varepsilon\left[\left[\frac{\partial u_1}{\partial x_o} + \frac{\partial u_1}{\partial y_o}\frac{\partial y_o}{\partial x}\right] + y_1\left[\frac{\partial^2 u_o}{\partial x_o \partial y_o} + \frac{\partial^2 u_o}{\partial y_o^2}\frac{\partial y_o}{\partial x}\right] + \frac{\partial u_o}{\partial y_o}\frac{\partial y_1}{\partial x}\right] \\
& + \frac{1}{2}\varepsilon^2 \Bigg[\left[\frac{\partial u_2}{\partial x_o} + \frac{\partial u_2}{\partial y_o}\frac{\partial y_o}{\partial x}\right] + 2y_1\left[\frac{\partial^2 u_1}{\partial x_o \partial y_o} + \frac{\partial^2 u_1}{\partial y_o^2}\frac{\partial y_o}{\partial x}\right] + 2\frac{\partial u_1}{\partial y_o}\frac{\partial y_1}{\partial x} \\
& + y_1^2\left[\frac{\partial^3 u_o}{\partial x_o \partial y_o^2} + \frac{\partial^3 u_o}{\partial y_o^3}\frac{\partial y_o}{\partial x}\right] + 2y_1\frac{\partial^2 u_o}{\partial y_o^2}\frac{\partial y_1}{\partial x} + y_2\left[\frac{\partial^2 u_o}{\partial x_o \partial y_o} + \frac{\partial^2 u_o}{\partial y_o^2}\frac{\partial y_o}{\partial x}\right] + \frac{\partial u_o}{\partial y_o}\frac{\partial y_2}{\partial x}\Bigg] + \ldots
\end{aligned}$$

where the chain rule has been used to differentiate $u_o, \partial u_o/\partial y_o, \partial^2 u_o/\partial y_o^2, u_1, \partial u_1/\partial y_o$ and u_2, but it has not been used to differentiate y_1 and y_2. Again, these derivatives are simply denoted $\partial y_1/\partial x$ and $\partial y_2/\partial x$.

The final formula for $\partial u/\partial x$ obtains on eliminating $\partial y_o/\partial x$ by the use of Equation (A.21) and it is

$$\frac{\partial u}{\partial x}=\frac{\partial u_o}{\partial x_o}+\varepsilon\left[\frac{\partial u_1}{\partial x_o}+y_1\frac{\partial^2 u_o}{\partial x_o\partial y_o}\right]$$
$$+\frac{1}{2}\varepsilon^2\left[\frac{\partial u_2}{\partial x_o}+2y_1\frac{\partial^2 u_1}{\partial x_o\partial y_o}+y_1^2\frac{\partial^3 u_o}{\partial x_o\partial y_o^2}+y_2\frac{\partial^2 u_o}{\partial x_o\partial y_o}\right]+\ldots \tag{A.22}$$

Again, the derivatives of $y_1, y_2, \ldots$ are lost in the last step and the formula for $\partial u/\partial x$, similar to the formula for $\partial u/\partial y$, takes the form of the formula for u itself. This, of course, makes it possible to guess other formulas *e.g.*, the formula for $\partial^2 u/\partial x^2$. It is worthwhile observing that each time $\partial y_o/\partial x$ is eliminated, by substituting $-\varepsilon\frac{\partial y_1}{\partial x}-\frac{1}{2}\varepsilon^2\frac{\partial y_2}{\partial x}-\ldots$ in its place, terms to all higher order in ε are introduced. It is by this that the derivatives of $y_1, y_2, \ldots$ cancel out; for example, just as $\frac{\partial u_o}{\partial y_o}\frac{\partial y_o}{\partial x}$ cancels $\varepsilon\frac{\partial u_o}{\partial y_o}\frac{\partial y_1}{\partial x}$ and $\frac{1}{2}\varepsilon^2\frac{\partial u_o}{\partial y_o}\frac{\partial y_2}{\partial x}$ in the forgoing calculations, so too it will cancel $\frac{1}{6}\varepsilon^3\frac{\partial u_o}{\partial y_o}\frac{\partial y_3}{\partial x}$ at the third order.

Equations (A.17), (A.19), (A.20) and (A.22) present for the expansion of a domain variable, yet two kinds of variables can appear in the boundary equations: domain variables evaluated at the boundary and variables defined only at the boundary.

The expansion of a domain variable evaluated at the boundary simply requires the mapping to be evaluated at the boundary in the formulas already obtained. The expansion of a variable defined only at the boundary stems from the expansion of the function determining the shape of the boundary. The function Y defines the shape of the boundary in the present configuration. Its expansion along the mapping appears as Equation (A.8).

Let n denote a function defined only at the boundary of the present domain. Now, n is typically a function of Y and we denote its values by $n(Y)$. The expansion of n, then, must come out of the expansion of Y, and to expand n in powers of ε along the mapping, we write

$$n(Y) = n(\varepsilon = 0) + \varepsilon \frac{dn}{d\varepsilon}(\varepsilon = 0) + \frac{1}{2}\varepsilon^2 \frac{d^2n}{d\varepsilon^2}(\varepsilon = 0) + \ldots$$

and calculate $\frac{dn}{d\varepsilon}, \frac{d^2n}{d\varepsilon^2}, \ldots$. As n does not depend explicitly on ε, the first derivative is[2]

$$\frac{dn}{d\varepsilon} = n_Y \left\{ \frac{dY}{d\varepsilon} \right\}$$

and the second derivative is

$$\frac{d^2n}{d\varepsilon^2} = \frac{d}{d\varepsilon}\frac{dn}{d\varepsilon} = n_Y \left\{ \frac{d^2Y}{d\varepsilon^2} \right\} + n_{YY} \left\{ \frac{dY}{d\varepsilon} \right\} \left\{ \frac{dY}{d\varepsilon} \right\}$$

where n_Y is a linear operator and n_{YY} is a bilinear operator. Both depend on Y. Using these formulas and

---

[2] As n is a function of Y, and Y is, itself, a function, some way of indicating the derivative of n with respect to Y is required. The notation $n_Y\{\ \}$ does this, where n_Y operates on whatever appears inside $\{\ \}$. In fact, n_Y can be obtained as

$$n_Y\{X\} = \lim_{\varepsilon \to 0} \frac{n(Y + \varepsilon X) - n(Y)}{\varepsilon}$$

where n_Y may again be a function of Y. It is a linear function of X, hence, the use of $\{\ \}$. In terms of n_Y, there obtains

$$n(Y + \varepsilon X) = n(Y) + \varepsilon n_Y\{X\} + R$$

where $R/\varepsilon \to 0$ as $\varepsilon \to 0$.

$$\frac{dY}{d\varepsilon} = Y_1 + \varepsilon Y_2 + \dots$$

$$\frac{d^2Y}{d\varepsilon^2} = Y_2 + \dots$$

$$\vdots$$

etc.

and evaluating the derivatives of n at $\varepsilon = 0$ produces the expansion of $n(Y)$.[3] It is, on writing $n(Y_o)$ in place of $n(\varepsilon = 0)$,

$$n(Y) = n(Y_o) + \varepsilon n_Y \{Y_1\} + \frac{1}{2}\varepsilon^2 \left[n_Y \{Y_2\} + n_{YY} \{Y_1\}\{Y_1\} \right] + \dots \qquad \text{(A.23)}$$

where n_Y and n_{YY} must be evaluated at Y_o.

Next, an example will be presented to help the reader understand how to apply this method.

A.1 Example of How to Use the Expansions of u and Its Derivatives

The most direct way to obtain the equations satisfied by u_o, u_1, u_2, etc. on the reference domain is to substitute the expansions of u and its derivatives into the equations that must be satisfied on the present domain. Then as the right-hand sides of

---

[3] In a definite problem, n must be specified as definite function of Y. Its expansion can then be obtained most easily by differentiating this specified function; for example, if $\vec{n}$ denotes the normal to the curve $y = Y(x, \varepsilon)$, then

$$\vec{n} = \frac{-\dfrac{\partial Y}{\partial x}\vec{i} + \vec{j}}{\sqrt{\left[\dfrac{\partial Y}{\partial x}\right]^2 + 1}}$$

and the derivatives of $\vec{n}$ can be obtained in terms of the derivatives of Y. But if n is not specified as a definite function of Y, the notation n_Y, n_{YY}, etc. must be introduced.

these expansions depend only on x_o, y_o and ε, on setting the coefficients of each power of ε to zero, a set of equations in u_o, u_1, u_2, etc. appear.

The next example will illustrate the method and it will mainly show that the mapping does not survive in the equations satisfied by u_o, u_1, u_2, etc. This is a favorable result because the mapping of the interior of the reference domain to the interior of the present domain cannot be determined. The derivatives of y_1, y_2, etc. were already lost in expanding the derivatives of u, so they cannot appear in the equations for u_o, u_1, u_2, etc.; now y_1, y_2, etc. themselves are lost.

To take an example, let u satisfy

$$u\frac{\partial u}{\partial y}+\varepsilon u+\varepsilon^2+1=0$$

on the current domain. Then substituting the expansions of $u(x,y,\varepsilon)$ and $\frac{\partial u}{\partial y}(x,y,\varepsilon)$ into this, there obtains

$$\begin{aligned}&\left[u_o+\varepsilon\left(u_1+y_1\frac{\partial u_o}{\partial y_o}\right)+\frac{1}{2}\varepsilon^2\left(u_2+2y_1\frac{\partial u_1}{\partial y_o}+y_1^2\frac{\partial^2 u_o}{\partial y_o^2}+y_2\frac{\partial u_o}{\partial y_o}\right)+\cdots\right]\\ &\times\left[\frac{\partial u_o}{\partial y_o}+\varepsilon\left(\frac{\partial u_1}{\partial y_o}+y_1\frac{\partial^2 u_o}{\partial y_o^2}\right)+\frac{1}{2}\varepsilon^2\left(\frac{\partial u_2}{\partial y_o}+2y_1\frac{\partial^2 u_1}{\partial y_o^2}+y_1^2\frac{\partial^3 u_o}{\partial y_o^3}+y_2\frac{\partial^2 u_o}{\partial y_o^2}\right)+\cdots\right]\\ &+\varepsilon\left[u_o+\varepsilon\left(u_1+y_1\frac{\partial u_o}{\partial y_o}\right)+\cdots\right]+\varepsilon^2+1=0\end{aligned}$$

which leads, on requiring the coefficient of each power of ε to vanish, to the following. First at order ε^o, which is the reference domain, the domain equation is

$$u_o\frac{\partial u_o}{\partial y_o}+1=0 \qquad \text{(A.24)}$$

For order ε^1, the domain equation is

$$u_o\left(\frac{\partial u_1}{\partial y_o}+y_1\frac{\partial^2 u_o}{\partial y_o^2}\right)+\left(u_1+y_1\frac{\partial u_o}{\partial y_o}\right)\frac{\partial u_o}{\partial y_o}+u_o=0$$

For order ε^2, the domain equation is

$$\frac{1}{2}u_o\left(\frac{\partial u_2}{\partial y_o}+2y_1\frac{\partial^2 u_1}{\partial y_o^2}+y_1^2\frac{\partial^3 u_o}{\partial y_o^3}+y_2\frac{\partial^2 u_o}{\partial y_o^2}\right)+\frac{1}{2}\left(u_2+2y_1\frac{\partial u_1}{\partial y_o}+y_1^2\frac{\partial^2 u_o}{\partial y_o^2}+y_2\frac{\partial u_o}{\partial y_o}\right)\frac{\partial u_o}{\partial y_o}$$
$$+\left(u_1+y_1\frac{\partial u_o}{\partial y_o}\right)\left(\frac{\partial u_1}{\partial y_o}+y_1\frac{\partial^2 u_o}{\partial y_o^2}\right)+\left(u_1+y_1\frac{\partial u_o}{\partial y_o}\right)+1=0$$

Now by using $\frac{\partial}{\partial y_o}\left(u_o\frac{\partial u_o}{\partial y_o}+1\right)=0$, the first equation, Equation (A.24) eliminates

$y_1\left(u_o\frac{\partial^2 u_o}{\partial y_o^2}+\frac{\partial u_o}{\partial y_o}\frac{\partial u_o}{\partial y_o}\right)$ in the second, reducing it to

$$u_o\frac{\partial u_1}{\partial y_o}+u_1\frac{\partial u_o}{\partial y_o}+u_o=0 \tag{A.25}$$

while the third equation can be written

$$\frac{1}{2}u_o\frac{\partial u_2}{\partial y_o}+\frac{1}{2}u_2\frac{\partial u_o}{\partial y_o}+u_1\frac{\partial u_1}{\partial y_o}+u_1+1+y_1\left(u_o\frac{\partial^2 u_1}{\partial y_o^2}+\frac{\partial u_1}{\partial y_o}\frac{\partial u_o}{\partial y_o}+\frac{\partial u_o}{\partial y_o}\frac{\partial u_1}{\partial y_o}+u_1\frac{\partial^2 u_o}{\partial y_o^2}+\frac{\partial u_o}{\partial y_o}\right)$$
$$+\frac{1}{2}y_1^2\left(u_o\frac{\partial^3 u_o}{\partial y_o^3}+\frac{\partial^2 u_o}{\partial y_o^2}\frac{\partial u_o}{\partial y_o}+2\frac{\partial u_o}{\partial y_o}\frac{\partial^2 u_o}{\partial y_o^2}\right)+\frac{1}{2}y_2\left(u_o\frac{\partial^2 u_o}{\partial y_o^2}+\frac{\partial u_o}{\partial y_o}\frac{\partial u_o}{\partial y_o}\right)=0$$

In this, the terms multiplying y_1 add to zero, as do the terms multiplying y_1^2 and those multiplying y_2. The sum of the terms multiplying y_1 vanishes on the use of

$$\frac{\partial}{\partial y_o}\left(u_o\frac{\partial u_1}{\partial y_o}+\frac{\partial u_o}{\partial y_o}u_1+u_o\right)=0$$

while the sums of terms multiplying y_1^2 and y_2 vanish on the use of

$$\frac{\partial^2}{\partial y_o^2}\left(u_o\frac{\partial u_o}{\partial y_o}+1\right)=0=\frac{\partial}{\partial y_o}\left(u_o\frac{\partial u_o}{\partial y_o}+1\right)$$

whence the third equation reduces to

$$\frac{1}{2}u_o\frac{\partial u_2}{\partial y_o}+\frac{1}{2}\frac{\partial u_o}{\partial y_o}u_2+u_1\frac{\partial u_1}{\partial y_o}+u_1+1=0 \qquad \text{(A.26)}$$

Again a sequence of equations on the reference domain is obtained, Equations (A.24), (A.25) and (A.26), and, again they are free of the mapping. The mapping is lost in writing the equations on the reference domain. The earlier equations in the sequence eliminate the mapping in the later equations. The result, then, is an equation for u_o at order zero. It is just the original equation for u written on the reference domain. Then comes an equation for u_1 in terms of u_o, succeeded by an equation for u_2 in terms of u_o and u_1, etc.

To completely determine u_o, u_1, u_2, etc., equations must be worked out at the boundary of the reference domain. In these the mapping of the boundary must remain if determining the present domain is to be part of the solution. Yet before deriving the equations at the boundary of the reference domain, a small matter remain to be explained before our work on the domain equations is complete. It is a rule by which the correct equations on the reference domain can be obtained without the need to remove the mapping at each step by using equations obtained in earlier steps. This rule is explained in the following section.

A.2 The Rule

The rule is this: To get the correct equations satisfied by u_o, u_1, u_2, etc. on the reference domain, replace u wherever it is present in the equations on the current domain by

$$u_o + \varepsilon u_1 + \frac{1}{2}\varepsilon^2 u_2 + \cdots$$

and then replace $\frac{\partial}{\partial y}$ by $\frac{\partial}{\partial y_o}$, $\frac{\partial}{\partial x}$ by $\frac{\partial}{\partial x_o}$, etc.

To see that this works, let u satisfy

$$u\frac{\partial u}{\partial y} + \varepsilon u + \varepsilon^2 + 1 = 0$$

on the current domain. Then, making the replacement there obtains

$$\left(u_o + \varepsilon u_1 + \frac{1}{2}\varepsilon^2 u_2 + \cdots\right)\left(\frac{\partial u_o}{\partial y_o} + \varepsilon\frac{\partial u_1}{\partial y_o} + \frac{1}{2}\varepsilon^2\frac{\partial u_2}{\partial y_o} + \cdots\right)$$
$$+\left(u_o + \varepsilon u_1 + \frac{1}{2}\varepsilon^2 u_2 + \cdots\right) + \varepsilon^2 + 1 = 0$$

and setting the coefficient of each power of ε to zero leads to

$$u_o\frac{\partial u_o}{\partial y_o} + 1 = 0$$

$$u_o\frac{\partial u_1}{\partial y_o} + u_1\frac{\partial u_o}{\partial y_o} + u_o = 0$$

$$\frac{1}{2}u_o\frac{\partial u_2}{\partial y_o} + \frac{1}{2}u_2\frac{\partial u_o}{\partial y_o} + u_1\frac{\partial u_1}{\partial y_o} + u_1 + 1 = 0$$

$$\vdots$$

etc.

This is the sequence of equations obtained earlier as Equations (A.24), (A.25) and (A.26), but, by use of the rule explained here, obtaining the sequence requires a lot less work.

Now to complete the sequence of problems that must be solved to obtain u_o, u_1, u_2, etc., we must add equations at the boundary to the equations on the reference domain. Again, to illustrate the method, we work out an example. Let u be a domain variable and let n be a boundary variable where the values of n along the boundary depend on the shape of the boundary. The dependence of n and Y must be specified, but Y may remain to be determined.

At the boundary of the present domain, let u and n satisfy

$$n(Y)\frac{\partial u}{\partial y}(x,Y,\varepsilon)=0 \tag{A.27}$$

where $y=Y(x,\varepsilon)$ defines the shape of the boundary, where the expansion of Y is

$$y=Y(x,\varepsilon)=Y_o(x_o)+\varepsilon Y_1(x_o)+\frac{1}{2}\varepsilon^2 Y_2(x_o)+\ldots$$

and where n inherits its ε dependence through Y. Then, substitute into Equation (A.27) the expansions of $n(Y)$ and $\frac{\partial u}{\partial y}(x,Y,\varepsilon)$.

The expansion of $\partial u/\partial y$ at the boundary can be written by evaluating its domain expansion, viz., Equation (A.19), at $y=Y(x,\varepsilon)$ and hence at $y_o=Y_o(x_o)$. It is

$$\frac{\partial u}{\partial y}(x,Y,\varepsilon)=\frac{\partial u_o}{\partial y_o}+\varepsilon\left[\frac{\partial u_1}{\partial y_o}+Y_1\frac{\partial^2 u_o}{\partial y_o^2}\right]+\frac{1}{2}\varepsilon^2\left[\frac{\partial u_2}{\partial y_o}+2Y_1\frac{\partial^2 u_1}{\partial y_o^2}+Y_1^2\frac{\partial^3 u_o}{\partial y_o^3}+Y_2\frac{\partial^2 u_o}{\partial y_o^2}\right]+\ldots$$

where all the variable on the right-hand side must be evaluated at $y_o = Y_o(x_o)$, and hence they all depend only on x_o, and where the right-hand side is already defined on the boundary of the reference domain. The expansion of $n(Y)$ has been obtained as Equation (A.23), namely

$$n(Y) = n(Y_o) + \varepsilon n_Y\{Y_1\} + \frac{1}{2}\varepsilon^2\left[n_Y\{Y_2\} + n_{YY}\{Y_1\}\{Y_1\}\right] + \ldots$$

where n_Y and n_{YY} depend also on Y_o. Substituting these expansions into Equation (A.27) then leads to a series of equations on the boundary of the reference domain, one at each order in ε. The sequence begins

$$n(Y_o)\frac{\partial u_o}{\partial y_o}(x_o, Y_o) = 0$$

and

$$n(Y_o)\left[\frac{\partial u_1}{\partial y_o}(x_o, Y_o) + Y_1\frac{\partial^2 u_o}{\partial y_o^2}(x_o, Y_o)\right] + n_Y\{Y_1\}\frac{\partial u_o}{\partial y_o}(x_o, Y_o) = 0$$

and so on. Each equation holds on the boundary of the reference domain, viz., on $y_o = Y_o(x_o)$, and the derivatives n_Y, n_{YY}, etc. must be evaluated at $Y = Y_o$. This often leads to simple formulas if Y_o is simple *e.g.*, if Y_o is a constant.

As Y_o must be known, the first equation completes the problem for u_o. Of course, the problem for u_o is just the nonlinear problem on the reference domain. Then, the second equation completes the problem for u_1, or at least it completes the problem for u_1 if Y_1 is known. Often Y_1 is not known and a second boundary equation must then be specified if Y_1 is to be determined. This leads to another equation in u_1 and Y_1 at the boundary of the reference domain and the two variables must be determined together, likewise u_2 and Y_2, etc. As the functions Y_1, Y_2, etc. become known, so too the shape of the present domain to first, second, etc. orders in ε.

It is important to observe that the derivation of the boundary equations is just like the derivation of the domain equations, save that the mapping cannot be eliminated from the boundary equations. It is important that the mapping be eliminated in the domain equations, for it cannot be determined on the domain; it is equally important that it cannot be eliminated from the boundary equations, otherwise the shape of the present domain could not be determined.

A.3 Expansion of Integrals in Mapping

For a cylindrical coordinate system, let $\vec{r}$ denote the position vector to the point (r,θ,z) in the present domain and let $\vec{r}_o$ denote the position vector to the point (r_o,θ_o,z_o) in the reference domain. Denote a mapping of the reference domain into the present domain by

$$\vec{r} = \vec{r}\left(\vec{r}_o,\varepsilon\right)$$

or by

$$r = r\left(r_o,\theta_o,z_o,\varepsilon\right)$$

$$\theta = \theta\left(r_o,\theta_o,z_o,\varepsilon\right)$$

and

$$z = z\left(r_o,\theta_o,z_o,\varepsilon\right)$$

Let the mapping be expanded in powers of ε as

$$\vec{r} = \vec{r}_o + \varepsilon \vec{r}_1 + \frac{1}{2}\varepsilon^2 \vec{r}_2 + \ldots$$

or as

$$r = r_o + \varepsilon r_1 + \frac{1}{2}\varepsilon^2 r_2 + \ldots$$

$$\theta = \theta_o + \varepsilon \theta_1 + \frac{1}{2}\varepsilon^2 \theta_2 + \ldots$$

and

$$z = z_o + \varepsilon z_1 + \frac{1}{2}\varepsilon^2 z_2 + \ldots$$

where $\vec{r}_1, r_1, \theta_1, z_1, \vec{r}_2, r_2, \theta_2, z_2, \ldots$ depend on $\vec{r}_o$ or on r_o, θ_o and z_o and where

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial \varepsilon}(\varepsilon = 0) \ , \quad \vec{r}_2 = \frac{\partial^2 \vec{r}}{\partial \varepsilon^2}(\varepsilon = 0) \ , \ldots$$

$$r_1 = \frac{\partial r}{\partial \varepsilon}(\varepsilon = 0) \ , \quad r_2 = \frac{\partial^2 r}{\partial \varepsilon^2}(\varepsilon = 0) \ , \ldots$$

$$\theta_1 = \frac{\partial \theta}{\partial \varepsilon}(\varepsilon = 0) \ , \quad \theta_2 = \frac{\partial^2 \theta}{\partial \varepsilon^2}(\varepsilon = 0) \ , \ldots$$

and

$$z_1 = \frac{\partial z}{\partial \varepsilon}(\varepsilon = 0) \ , \quad z_2 = \frac{\partial^2 z}{\partial \varepsilon^2}(\varepsilon = 0) \ , \ldots$$

Now let V denote the volume of the present domain. The, in many cases, there are physical requirements on some state variable, denoted by u, which can be stated in terms of its integral over V. In this appendix, the form of this requirement is determined to first order in ε.

Let I denote the integral of u over V, namely

$$I = \int_V u\,dV$$

Now, u depends on $\vec{r}$ and ε and V depends on ε; hence, I depends on ε. The mapping $\vec{r} = \vec{r}_o + \varepsilon\vec{r}_1 + \ldots$ carries the points $\vec{r}_o$ of the reference domain into the points $\vec{r}$ of the present domain, and via this, it ties V to V_o.

Our job is to expand I along the mapping in powers of ε. To do this write

$$I = I_o + \varepsilon I_1 + \ldots$$

where

$$I_o = I(\varepsilon = 0)$$

$$I_1 = \frac{dI}{d\varepsilon}(\varepsilon = 0)$$

$$\vdots$$

etc.

and where

$$I_o = \int_{V_o} u_o\,dV_o$$

To determine I_1, use Leibnitz' rule to write

$$\frac{dI}{d\varepsilon} = \int_V \frac{\partial u}{\partial \varepsilon}dV + \int_{\partial V} dA\vec{n}.\frac{d\vec{r}}{d\varepsilon}u$$

Then, set $\varepsilon = 0$ and use $\frac{d\vec{r}}{d\varepsilon}(\varepsilon = 0) = \vec{r}_1$, where, at the boundary of the reference configuration, $\vec{r}_1 = \vec{R}_1$, to obtain

$$I_1 = \frac{dI}{d\varepsilon}(\varepsilon = 0) = \int_{V_o} u_1\,dV_o + \int_{\partial V_o} dA_o\vec{n}_o.\vec{R}_1 u_o$$

and, by this, the expansion of I to two terms, namely

$$I = \int_{V_o} u_o dV_o + \varepsilon\left[\int_{V_o} u_1 dV_o + \int_{\partial V_o} dA_o \vec{n}_o . \vec{R}_1 u_o\right] \tag{A.28}$$

Now, the mapping of the domain does not appear in Equation (A.28); only the mapping of the boundary appears via $\vec{R}_1$. To see how the mapping is lost, the calculation can be carried out another way. To do this, let u depend on x, y and ε and let the mapping be written

$$x = x_o$$

and

$$y = y_o + \varepsilon y_1(x_o, y_o) + \ldots$$

Then, to evaluate $\iint_V u dx dy$, substitute the expansion of u along the mapping and at the same time transform the integral over the present domain to an integral over the reference domain to get

$$\iint_V u(x, y, \varepsilon) dx dy = \iint_{V_o}\left[u_o + \varepsilon\left(u_1 + y_1 \frac{\partial u_o}{\partial y_o}\right) + \ldots\right] \frac{\partial(x, y)}{\partial(x_o, y_o)} dx_o dy_o$$

where

$$\frac{\partial(x, y)}{\partial(x_o, y_o)} = 1 + \varepsilon \frac{\partial y_1}{\partial y_o} + \ldots$$

To first order in ε, this is

$$\iint_V u(x, y, \varepsilon) dx dy = \iint_{V_o} u_o dx_o dy_o + \varepsilon \iint_{V_o}\left[u_1 + \frac{\partial(y_1 u_o)}{\partial y_o}\right] dx_o dy_o$$

and, carrying out the last integration on the right-hand side, there obtains

$$\iint_{V} u(x,y,\varepsilon)\,dxdy = \iint_{V_o} u_o dx_o dy_o + \varepsilon\left[\iint_{V_o} u_1 dx_o dy_o + \int_{y_o=Y(x_o)} Y_1 u_o dx_o\right] \qquad \text{(A.29)}$$

The mapping of the domain does not appear in this formula; it is lost in the last step, leaving only the mapping of the boundary. Now, let $u=1$, and hence $u_o=1, u_1=0$, etc.; then Equation (A.29) reduces to

$$\iint_{V} dxdy = \iint_{V_o} dx_o dy_o + \varepsilon \int_{y_o=Y(x_o)} Y_1 dx_o$$

This is a formula for the volume (area) of a perturbed domain.

APPENDIX B
OVERVIEW OF METHODS USED IN CHAPTER 3

The following chart, Figure B.1, gives a clear idea about the methods used through out Chapter 3. It describes the calculation steps for both the steady state problem and for the eigenvalue problem. It also summarizes the critical points for each case that we found in the chapter. It shows how one might change the conditions of a given case to move to another one without losing the ability to get the correct critical point.

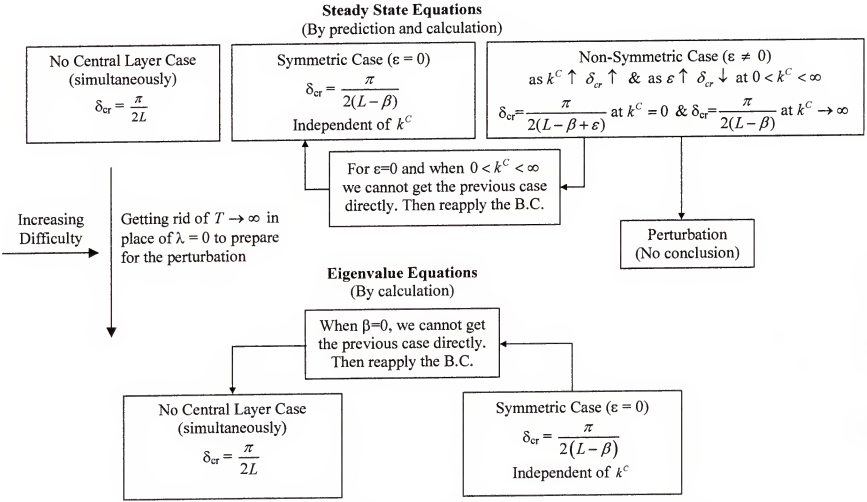
Steady State Equations
(By prediction and calculation)
No Central Layer Case
(simultaneously)
$\delta_{cr} = \frac{\pi}{2L}$
Symmetric Case ($\varepsilon = 0$)
$\delta_{cr} = \frac{\pi}{2(L-\beta)}$
Independent of k^C
Non-Symmetric Case ($\varepsilon \neq 0$)
as $k^C \uparrow$ $\delta_{cr} \uparrow$ & as $\varepsilon \uparrow$ $\delta_{cr} \downarrow$ at $0 < k^C < \infty$
$\delta_{cr} = \frac{\pi}{2(L-\beta+\varepsilon)}$ at $k^C = 0$ & $\delta_{cr} = \frac{\pi}{2(L-\beta)}$ at $k^C \to \infty$
For ε=0 and when $0 < k^C < \infty$ we cannot get the previous case directly. Then reapply the B.C.
Perturbation
(No conclusion)
Increasing
Difficulty
Getting rid of $T \to \infty$ in place of $\lambda = 0$ to prepare for the perturbation
Eigenvalue Equations
(By calculation)
When β=0, we cannot get the previous case directly. Then reapply the B.C.
No Central Layer Case
(simultaneously)
$\delta_{cr} = \frac{\pi}{2L}$
Symmetric Case ($\varepsilon = 0$)
$\delta_{cr} = \frac{\pi}{2(L-\beta)}$
Independent of k^C

Figure B-1. Chart of the main methods used in Chapter 3 to test for the thermal stability.

APPENDIX C
LOCATING THE MAXIMUM TEMPERATURE IN THE CASE OF CONSTANT TEMPERATURE WALLS

This appendix gives an explanation about why it is not possible to guess the exact location of the maximum temperature in the case of constant temperature walls. As usual, this explanation is based on solving a simple problem. This simple problem is just the isothermal problem considered in section 4.2.1.1 in Chapter 4 except that the heat source is taken to be independent of temperature. This problem is depicted in Figure C-1. It will be of interest to find the location of the maximum temperature.

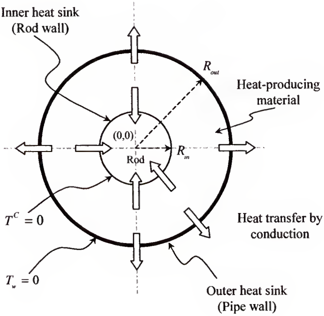

Figure C-1. Heat transfer in the case of constant temperature walls in a perfectly centered annulus.

For this geometry the temperature rise at steady state, referring to Figure C-1 for a definition of the variables, can be described by the thermal energy equation

$$\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}+Q=0 \tag{C.1}$$

where T is the temperature field and where Q denotes the rate at which heat is produced in the material per unit area. The temperature is subject to two boundary conditions viz.,

$$T\left(r=R_{out}\right)=0 \tag{C.2}$$

and

$$T\left(r=R_{in}\right)=0 \tag{C.3}$$

By solving Equation (C.1) we get

$$T=C_1\ln(r)+C_2-\frac{Qr^2}{4} \tag{C.4}$$

where C_1 and C_2 are arbitrary constants which can be determined by using Equations (C.2) and (C.3) viz.,

$$C_1\ln(R_{out})+C_2-\frac{QR_{out}^2}{4}=0 \tag{C.5}$$

and

$$C_1\ln(R_{in})+C_2-\frac{QR_{in}^2}{4}=0 \tag{C.6}$$

Hence, by solving these two equations we get

$$C_1=\frac{Q\left(R_{out}^2-R_{in}^2\right)}{4\ln\left(\dfrac{R_{out}}{R_{in}}\right)}$$

and

$$C_2 = \frac{Q}{4}\left[R_{out}^2 - \frac{\left(R_{out}^2 - R_{in}^2\right)}{\ln\left(\frac{R_{out}}{R_{in}}\right)}\ln\left(R_{out}\right)\right]$$

Thus, Equation (C.4) becomes

$$T = \frac{Q\left(R_{out}^2 - R_{in}^2\right)}{4\ln\left(\frac{R_{out}}{R_{in}}\right)}\ln(r) + \frac{Q}{4}\left[R_{out}^2 - \frac{\left(R_{out}^2 - R_{in}^2\right)}{\ln\left(\frac{R_{out}}{R_{in}}\right)}\ln\left(R_{out}\right)\right] - \frac{Qr^2}{4} \tag{C.7}$$

But the maximum temperature is found when $dT/dr = 0$, which means

$$\frac{dT}{dr} = \frac{Q\left(R_{out}^2 - R_{in}^2\right)}{4\ln\left(\frac{R_{out}}{R_{in}}\right)r} - \frac{Qr}{2} = 0 \tag{C.8}$$

where r is the radius at which the maximum temperature is located, $r\,\Big|_{T=T_{\max}}$. Thus, we can write

$$r^2\Big|_{T=T_{\max}} = \frac{\left(R_{out}^2 - R_{in}^2\right)}{2\ln\left(\frac{R_{out}}{R_{in}}\right)} \tag{C.9}$$

From this we find that the maximum temperature is not located at the middle of the annular region as someone might guessed because $r\,\Big|_{T=T_{\max}}$ is not equal to the arithmetic radius or mean[1]

$$r\Big|_{arithmetic\ mean} = \frac{\left(R_{out} + R_{in}\right)}{2} \tag{C.10}$$

---

[1] Also known as the linear mean, the average mean or the mean.

Therefore, the maximum temperature might be located either closer to the inner rod or closer to the outer pipe but not at an equal distance from the two. Then, to see whether the maximum temperature lies closer to the inner wall or the outer wall let us first rewrite Equation (C.9) in the following form

$$r^2\Big|_{T=T_{\max}} = \frac{\left(R_{out}+R_{in}\right)}{2}\frac{\left(R_{out}-R_{in}\right)}{\ln\left(\frac{R_{out}}{R_{in}}\right)} \tag{C.11}$$

and notice that it is just the arithmetic mean, which is defined by Equation (C.10), multiplied by the logarithmic radius or mean[2], which is

$$r\Big|_{logarithmic\ mean} = \frac{\left(R_{out}-R_{in}\right)}{ln\left(\frac{R_{out}}{R_{in}}\right)} \tag{C.12}$$

Then subtract the square of the arithmetic radius, $\left(r\big|_{arithmetic\ mean}\right)^2$, given by Equation (C.10) form Equation (C.11) viz.,

$$r^2\Big|_{T=T_{\max}} - \left(r\Big|_{arithmetic\ mean}\right)^2 = \frac{\left(R_{out}+R_{in}\right)}{2}\frac{\left(R_{out}-R_{in}\right)}{\ln\left(\frac{R_{out}}{R_{in}}\right)} - \left[\frac{\left(R_{out}+R_{in}\right)}{2}\right]^2 \tag{C.13}$$

or

$$r^2\Big|_{T=T_{max}} - \left[\frac{\left(R_{out}+R_{in}\right)}{2}\right]^2 = \underbrace{\frac{\left(R_{out}+R_{in}\right)}{2}}_{arithmetic\ mean}\left[\underbrace{\frac{\left(R_{out}-R_{in}\right)}{ln\left(\frac{R_{out}}{R_{in}}\right)}}_{logarithmic\ mean} - \underbrace{\frac{\left(R_{out}+R_{in}\right)}{2}}_{arithmetic\ mean}\right] \tag{C.14}$$

and on dividing by R_{in}^2 it becomes

---

[2] Also known as the log mean.

$$\frac{1}{R_{in}^2}\left(r^2\Big|_{T=T_{max}} - \left[\frac{\left(R_{out}+R_{in}\right)}{2}\right]^2 \right) = \underbrace{\frac{\left(\dfrac{R_{out}}{R_{in}}+1\right)}{2}}_{arithmetic\ mean}\left[\underbrace{\frac{\left(\dfrac{R_{out}}{R_{in}}-1\right)}{ln\left(\dfrac{R_{out}}{R_{in}}\right)}}_{logarithmic\ mean} - \underbrace{\frac{\left(\dfrac{R_{out}}{R_{in}}+1\right)}{2}}_{arithmetic\ mean}\right] \quad (C.15)$$

Now, if the sign of the right-hand side of Equation (C.15) is negative then $r^2\Big|_{T=T_{max}} < r^2\Big|_{arithmetic\ mean}$ and then of the maximum temperature is located somewhere in the middle of the region closer to the rod wall but if the sign of the right-hand side is positive then $r^2\Big|_{T=T_{max}} > r^2\Big|_{arithmetic\ mean}$ and consequently the maximum temperature is located closer to the pipe wall. To begin, notice that the right-hand side of Equation (C.15) is just the arithmetic mean multiplied by the difference between the logarithmic mean and the arithmetic mean. But we know that the logarithmic mean is always less than the arithmetic mean and to show this let us divide the arithmetic mean by the logarithmic mean which gives

$$\frac{Arithmetic\ mean}{Logarthmic\ mean} = \frac{\left(\dfrac{R_{out}}{R_{in}}+1\right)}{2\left(\dfrac{R_{out}}{R_{in}}-1\right)} ln\left(\frac{R_{out}}{R_{in}}\right) \quad (C.16)$$

then for any value of $R_{out}/R_{in} > 1$ Equation (C.16) is greater than one. From this we understand that the arithmetic mean is greater than the logarithmic mean. Based on this the right-hand side must be negative. Therefore, we conclude that the maximum temperature is located closer to the inner wall. From this result we can see that it is not possible to decide about the location of the maximum temperature in the case of constants temperature walls without doing some calculation.

APPENDIX D
EXPANSION OF THE UNIT OUTWARD NORMAL

The expansion of the unit outward normal, $\vec{n}$, can be written as

$$\vec{n} = \vec{n}_o + \varepsilon \vec{n}_1 + \frac{\varepsilon^2}{2} \vec{n}_2 + \ldots \tag{D.1}$$

where

$$\vec{n}_o = \vec{i}_r \tag{D.2}$$

and where $\vec{n}_1$ and $\vec{n}_2$ can be found by using equations

$$R^2 - 2\varepsilon R \cos\theta = R_{in}^2 - \varepsilon^2 \tag{D.3}$$

and

$$\vec{n} = \frac{\vec{i}_r - \vec{i}_\theta \dfrac{R'}{R}}{\sqrt{1 + \left(\dfrac{R'}{R}\right)^2}} \tag{D.4}$$

where R is a function of θ and ε and where

$$R' = \frac{\partial R(\theta)}{\partial \theta} \tag{D.5}$$

To do this, let us obtain R' by differentiating Equation (D.3) with respect to θ, viz.,

$$R' = -\frac{\varepsilon R \sin\theta}{R - \varepsilon \cos\theta} \tag{D.6}$$

Then, substitute this into Equation (D.4) to get

$$\vec{n} = \frac{\vec{i}_r + \vec{i}_\theta \dfrac{\varepsilon \sin\theta}{(R - \varepsilon\cos\theta)}}{\sqrt{1 + \dfrac{\varepsilon^2 \sin^2\theta}{(R - \varepsilon\cos\theta)^2}}} \tag{D.7}$$

which can be arranged as

$$\vec{n} = \frac{\vec{i}_r (R - \varepsilon\cos\theta) + \vec{i}_\theta \varepsilon \sin\theta}{\sqrt{(R - \varepsilon\cos\theta)^2 + \varepsilon^2 \sin^2\theta}} \tag{D.8}$$

and by expanding $(R - \varepsilon\cos\theta)^2$ we obtain

$$\vec{n} = \frac{\vec{i}_r (R - \varepsilon\cos\theta) + \vec{i}_\theta \varepsilon \sin\theta}{\sqrt{R^2 - 2\varepsilon R\cos\theta + \varepsilon^2}} \tag{D.9}$$

From Equation (D.3) we can see that the denominator of Equation (D.9) is just R_{in}.

Then, by substituting the expansion of R, which is

$$R = R_{in} + \varepsilon\cos\theta_o - \frac{\varepsilon^2}{2}\frac{\sin^2\theta_o}{R_{in}} + \ldots \tag{D.10}$$

into Equation (D.9) we get

$$\vec{n} = \frac{\vec{i}_r \left(R_{in} - \dfrac{\varepsilon^2}{2}\dfrac{\sin^2\theta_o}{R_{in}} + \ldots \right) + \vec{i}_\theta \varepsilon \sin\theta_o}{R_{in}} \tag{D.11}$$

From this we obtain

$$\vec{n}_1 = \vec{i}_\theta \frac{\sin\theta_o}{R_{in}} \tag{D.12}$$

and

$$\vec{n}_2 = -\vec{i}_r \frac{\sin^2\theta_o}{2R_{in}^2} \tag{D.13}$$

APPENDIX E
HEAT TRANSFER THROUGH A ROD FROM A WELL-MIXED TANK

Let us begin by describing the physics of this problem. To do this, consider Figure E-1 which depicts a rod of length L and of uniform cross-sectional area, A_{rod}, attached to a well-stirred tank. Initially (*i.e.*, at $t = 0$), the temperature throughout the tank is denoted by T^H. This is also the temperature at the end of the rod attached to the tank. The temperature at the other end of the rod (*i.e.*, the free end), is maintained at zero for all times. This is the heat sink. For $t > 0$, the tank will gain or lose heat by contact with the rod whose temperature is denoted $T(t, z)$. The temperature of the material in the tank is uniform through out the tank but it can change in time.

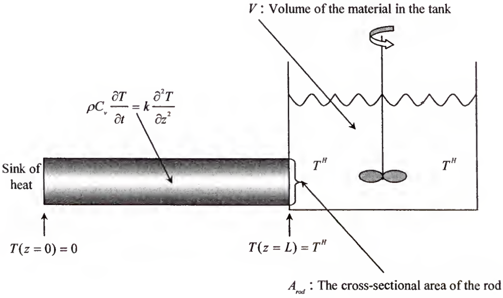
V: Volume of the material in the tank
$\rho C_v \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2}$
Sink of heat
T^H
T^H
$T(z = 0) = 0$
$T(z = L) = T^H$
A_{rod}: The cross-sectional area of the rod

Figure E-1. Heat transfer problem with a heat reservoir slowing the thermal diffusion.

Now, the temperature rise in the domain (*i.e.*, the rod), can be described by the thermal energy equation

$$\rho C_v \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2} \text{ , at } 0 < z < L \tag{E.1}$$

where ρ and C_v denote the density of the material of the rod and its heat capacity, respectively. In this problem, ρ and C_v are assumed to be constant. This equation is subject to two boundary conditions viz.,

$$T(z=0) = 0 \tag{E.2}$$

and

$$T(z=L) = T^H \tag{E.3}$$

where T^H is unknown and time dependent.

As T^H as well as T must be determined, we need another equation. This can be obtained via an energy balance about the tank, viz., by writing

$$-A_{rod} k \left. \frac{\partial T}{\partial z} \right|_{z=L} = V \rho^T C_v^T \frac{dT^H}{dt} \tag{E.4}$$

where V is the volume of the material in the tank, ρ^T is the density of this material and C_v^T is its heat capacity.

Equations (E.1) and (E.4) can be written in scaled form by defining $T = \overline{T}\ T^*$, $z = \overline{z}\ z^*$ and $t = \overline{t}\ t^*$ where $\overline{T}$ is selected to be equal to the ambient temperature and where $\overline{z} = L$ and $\overline{t} = L^2 \rho C_v / k$ and after dropping the *'s, we get

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial z^2} \tag{E.5}$$

where

$$T(z=0)=0 \tag{E.6}$$

and

$$T(z=1)=T^H \tag{E.7}$$

and where

$$\left.\frac{\partial T}{\partial z}\right|_{z=1}=-\frac{V\rho^T C_v^T}{A_{rod}L\rho C_v}\frac{dT^H}{dt} \tag{E.8}$$

Now to get the eigenvalue problem let us apply the separation of variables method to Equation (E.5). To do this, write T as

$$T(t,z)=\tau(t)\xi(z)$$

and substitute it into Equation (E.5). This will then give two ordinary differential equations, they are

$$\frac{d\tau}{dt}+\lambda^2\tau=0 \tag{E.9}$$

and

$$\frac{d^2\xi}{dz^2}+\lambda^2\xi=0 \tag{E.10}$$

where $-\lambda^2$ is the separation constant and where

$$\xi(z=0)=0 \tag{E.11}$$

and to get the remaining boundary condition in term of ξ substitute

$$T(t,z)=\tau(t)\xi(z)$$

into Equations (E.7) and (E.8) at $z=1$ and use Equation (E.9) to find

$$\left.\frac{d\xi}{dz}\right|_{z=1}=\left.\frac{V\rho^T C_v^T}{A_{rod}L\rho C_v}\lambda^2\xi\right|_{z=1} \tag{E.12}$$

Now, Equation (E.9) can be solved to give

$$\tau(t) = Ce^{-\lambda^2 t} \tag{E.13}$$

where C is an integration constant, where $e^{-\lambda^2 t}$ is the growth rate and where $-\lambda^2$ is the growth constant. Equation (E.10) is the eigenvalue problem that describes the temperature rise in the rod and its solution can be written as

$$\xi(z) = A\cos\lambda z + B\sin\lambda z \tag{E.14}$$

where A and B are arbitrary constants. Now A, B and λ remain to be determined. To do so we turn to the boundary conditions.

Now let us stop for a moment and assume that the well-mixed tank does not exist in the problem and further assume that the two ends of the rod are kept at fixed temperatures (*i.e.*, the two ends are isothermal). Then, the boundary conditions in this case are

$$\xi(z=0) = 0 \tag{E.15}$$

and

$$\xi(z=1) = 0 \tag{E.16}$$

and by applying these two boundary conditions into Equation (E.14) we find

$$A = 0 \tag{E.17}$$

and

$$B\sin\lambda = 0 \tag{E.18}$$

and from this we learn that $\lambda = n\pi$ and, consequently, the eigenvalue $-\lambda^2$ is

$$-\lambda^2 = -n^2\pi^2 \tag{E.19}$$

and for $n = 1$ it is

$$-\lambda^2 = -\pi^2 \tag{E.20}$$

But if we assumed that only one end of the rod is held at a fixed temperature, which is at $z=0$, while the other end is insulated (*i.e.*, one end is isothermal and one end is adiabatic), then the boundary conditions will become

$$\xi(z=0)=0 \tag{E.21}$$

and

$$\frac{d\xi}{dz}(z=1)=0 \tag{E.22}$$

and by applying these two boundary conditions into Equation (E.14) we get

$$A=0 \tag{E.23}$$

and

$$\lambda B\cos\lambda=0 \tag{E.24}$$

and from this we find that $\lambda=n\pi/2$ and, consequently, the eigenvalue $-\lambda^2$ will be

$$-\lambda^2=-\frac{n^2\pi^2}{4} \tag{E.25}$$

and for $n=1$ it is

$$-\lambda^2=-\frac{\pi^2}{4} \tag{E.26}$$

Now, by returning to the original problem where the well-mixed tank is attached to the rod at $z=1$, we have two boundary conditions given by Equations (E.11) and (E.12). Then, by substituting Equation (E.14) into these two boundary conditions, namely

$$\xi(z=0)=0$$

and

$$\left.\frac{d\xi}{dz}\right|_{z=1}=\left.\frac{V\rho^T C_v^T}{A_{rod}L\rho C_v}\lambda^2\xi\right|_{z=1}$$

and writing $\frac{V\rho^T C_v^T}{A_{rod} L \rho C_v}$ as $\frac{1}{\gamma}$ henceforth where γ is a physical property assumed known, we find

$$A = 0 \tag{E.27}$$

and

$$\lambda B \cos\lambda = \frac{1}{\gamma}\lambda^2 B \sin\lambda \tag{E.28}$$

or

$$\gamma = \lambda \tan\lambda \tag{E.29}$$

Hence A is zero, the allowable λ's are the roots of Equation (E.29) and B remains to be determined. It depends on the initial state of the system.

Now, let us rewrite Equation (E.29) as

$$\tan\lambda = \frac{\gamma}{\lambda} \tag{E.30}$$

The term γ is a positive constant and for simplicity it is assumed to be one. Thus, to find the allowable λ's we must solve

$$\tan\lambda = \frac{1}{\lambda} \tag{E.31}$$

The solution can be found by plotting graphs of $\tan\lambda$ and $1/\lambda$ *vs.* λ. This is shown in Figure E-2 where the solutions are the intersection. We number the solutions 1,2,3,... and give the first few in Table E-1.

Table E-1. Values of λ_n obtained from Figure E-2.

n	λ_n	n	λ_n
1	0.86	4	9.53
2	3.42	5	12.65
3	6.44	6	15.77

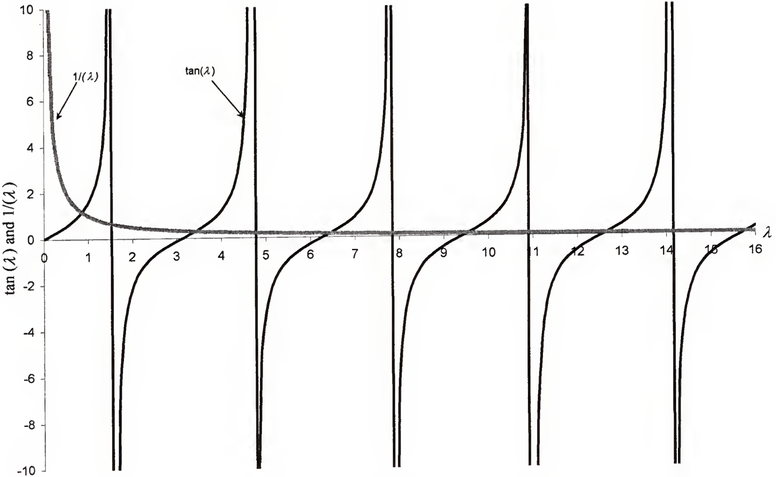

Figure E-2. $\tan\lambda$ and $1/\lambda$ *vs.* λ.

Now, let us ask the following question: in which of the three cases that we studied (*i.e.*, the rod with isothermal-isothermal ends, isothermal-adiabatic ends and isothermal-perfectly conducting ends), can the temperature perturbation die faster? To answer this question, let us compare the values of the growth constant, $-\lambda^2$, obtained in each case. For the rod with isothermal-isothermal ends, we get

$$-\lambda^2 = -\pi^2 = -9.87 \text{ , first case} \tag{E.32}$$

and for the rod with isothermal-adiabatic ends, we have

$$-\lambda^2 = -\frac{\pi^2}{4} = -2.47 \text{ , second case} \tag{E.33}$$

and for the rod with isothermal-perfectly conducting ends, we find

$$-\lambda^2 = -\frac{\pi^2}{4} = -0.74 \text{ , third case} \tag{E.34}$$

From this three values of the growth constant, $-\lambda^2$, we see that the temperature perturbation in the first case, the rod with isothermal-isothermal ends, will die faster than the other two cases. This is because in the first case the two isothermal ends serves as heat sinks and, therefore, the rod can loss the heat faster than the second case, the rod with isothermal-adiabatic ends, which has only one heat sink. In the third case, which is the slowest of them all, there is one heat sink as in the second case but now the well-mixed tank will gain some heat through the perfectly conducting end, but the tank has nowhere to loss this heat except through the rod. Thus, the tank will store the gained heat until the temperature in the rod become lower than it is in the tanks and then the tank will loss the heat to the rod, and this is why this case is slower than the second case, the case with adiabatic end.

Now, back to the well-mixed tank problem, with the values of λ_n known, the next step is to determine the values of the constants B_n in the solution, namely

$$T(t,z)=\sum_{n=1}^{\infty}e^{-\lambda_n^2 t}\left(B_n\sin\lambda_n z\right) \tag{E.35}$$

To do this, the initial condition (the temperature in the rod at $t=0$), must be expanded as

$$T(t=0,z)=\sum_{n=1}^{\infty}B_n\sin\lambda_n z=B_1\sin\lambda_1 z+B_2\sin\lambda_2 z+B_3\sin\lambda_3 z+\ldots \tag{E.36}$$

As an example, let the initial temperature be

$$T\left(t=0,z\right)=z\left(1-z\right)$$

then

$$z(1-z)\simeq B_1\sin\lambda_1 z+B_2\sin\lambda_2 z+B_3\sin\lambda_3 z+\ldots \tag{E.37}$$

To obtain values of the B_n's we might begin by multiplying Equation (E.37) by $\sin\lambda_n z$ and integrating both sides with respect to z over $\left(0,1\right)$ viz.,

$$\begin{aligned}\int_0^1 z(1-z)\sin\lambda_n z dz \simeq \int_0^1 B_1\sin\lambda_n z\sin\lambda_1 z dz\\ +\int_0^1 B_2\sin\lambda_n z\sin\lambda_2 z dz\\ +\int_0^1 B_3\sin\lambda_n z\sin\lambda_3 z dz+\ldots\end{aligned} \tag{E.38}$$

Doing this for each λ_n's, we get infinitely many equations in the infinitely many unknown B_n's and what is interesting here is that these equations do not simplify, as one might expect in simple heat transfer problem, as the functions $\sin\lambda_n z$ and $\sin\lambda_m z$ are not orthogonal in the plain vanilla inner product. So to proceed we must develop a sequence of approximations and the easiest way to do this is to consider a sequence of terminations (*i.e.*, first get B_1 assuming $B_2, B_3, \ldots$ are zero, then get B_1 and B_2, assuming $B_3, B_4, \ldots$ are zero, then get B_1, B_2 and B_3, assuming $B_4, B_5 \ldots$ are zero etc). The hope is, without going to far along, B_1 stabilizes, then B_2, etc. To see what happens, let us first get a first estimate of B_1 and B_2. To do this, let us multiply Equation (E.37) by $\sin \lambda_1 z$ and then integrate it with respect to z over $(0,1)$, this gives

$$\int_0^1 z(1-z)\sin \lambda_1 z dz \simeq \int_0^1 B_1 \sin \lambda_1 z \sin \lambda_1 z dz + \int_0^1 B_2 \sin \lambda_1 z \sin \lambda_2 z dz \tag{E.39}$$

Then, multiply Equation (E.37) by $\sin \lambda_2 z$ and then integrate it with respect to z over $(0,1)$, this gives

$$\int_0^1 z(1-z)\sin \lambda_2 z dz \simeq \int_0^1 B_1 \sin \lambda_2 z \sin \lambda_1 z dz + \int_0^1 B_2 \sin \lambda_2 z \sin \lambda_2 z dz \tag{E.40}$$

As we can see, Equations (E.39) and (E.40) are two equations in two unknowns (*i.e.*, B_1 and B_2), and, therefore, the values of B_1 and B_2 can be obtained. By using the sequence of termination technique we found that the values of B_1 and B_2 start to stabilize as the value of B_6 is obtained and Table E-2 can give a clear idea about how this technique is converging on the true values of B_n's.

Table E-2. Obtained values of B_1 to B_6.

The obtained B_n's	B_1	B_2	B_3	B_4	B_5	B_6
B_1, B_2	0.105	0.214	0	0	0	0
B_1, B_2, B_3	0.103	0.215	-0.001	0	0	0
B_1, B_2, B_3, B_4	0.092	0.221	-0.004	0.010	0	0
B_1, B_2, B_3, B_4, B_5	0.090	0.223	-0.005	0.010	-4×10^{-5}	0
$B_1, B_2, B_3, B_4, B_5, B_6$	0.089	0.224	-0.006	0.011	-5×10^{-4}	0.002

However, if we know how to write the correct inner product for this problem, then we can get the values of B_n's much easier. To do this, let us write our eigenvalue problem, Equation (E.10), in terms of the eigenfunction ξ_1 and the eigenvalue λ_1^2, which gives

$$\frac{d^2\xi_1}{dz^2}+\lambda_1^2\xi_1=0 \tag{E.41}$$

where

$$\xi_1(z=0)=0 \tag{E.42}$$

and

$$\left.\frac{d\xi_1}{dz}\right|_{z=1}=\left.\frac{1}{\gamma}\lambda_1^2\xi_1\right|_{z=1} \tag{E.43}$$

Then, rewrite Equation (E.10) in terms of the eigenfunction ξ_2 and the eigenvalue λ_2^2, which gives

$$\frac{d^2\xi_2}{dz^2}+\lambda_2^2\xi_2=0 \tag{E.44}$$

where

$$\xi_2(z=0)=0 \tag{E.45}$$

and

$$\left.\frac{d\xi_2}{dz}\right|_{z=1}=\left.\frac{1}{\gamma}\lambda_2^2\xi_2\right|_{z=1} \tag{E.46}$$

Now, multiply Equation (E.41) by ξ_2, to get

$$\xi_2\frac{d^2\xi_1}{dz^2}+\lambda_1^2\xi_2\xi_1=0 \tag{E.47}$$

and multiply Equation (E.44) by ξ_1, to obtain

$$\xi_1 \frac{d^2\xi_2}{dz^2} + \lambda_2^2 \xi_1 \xi_2 = 0 \tag{E.48}$$

Then, subtract Equation (E.48) from Equation (E.47), which gives

$$\xi_2 \frac{d^2\xi_1}{dz^2} - \xi_1 \frac{d^2\xi_2}{dz^2} = \left(\lambda_2^2 - \lambda_1^2\right)\xi_1\xi_2 \tag{E.49}$$

Now, integrate Equation (E.49) over the length of the rod, to get

$$\int_0^1 \left[\xi_2 \frac{d^2\xi_1}{dz^2} - \xi_1 \frac{d^2\xi_2}{dz^2}\right] dz = \left(\lambda_2^2 - \lambda_1^2\right) \int_0^1 \xi_1\xi_2 dz \tag{E.50}$$

and by using the integration by parts technique, we found

$$\xi_2 \frac{d\xi_1}{dz}\bigg|_0^1 - \xi_1 \frac{d\xi_2}{dz}\bigg|_0^1 = \left(\lambda_2^2 - \lambda_1^2\right) \int_0^1 \xi_1\xi_2 dz \tag{E.51}$$

Because $\xi_1 = 0$ and $\xi_2 = 0$ at $z = 0$ and by using Equations (E.43) and (E.46), Equation (E.51) reduces to

$$\frac{1}{\gamma}\left(\lambda_1^2 - \lambda_2^2\right)\xi_1\xi_2\big|_{z=1} = \left(\lambda_2^2 - \lambda_1^2\right) \int_0^1 \xi_1\xi_2 dz \tag{E.52}$$

which can be simplified and written as

$$\left(\lambda_2^2 - \lambda_1^2\right)\left[\int_0^1 \xi_1\xi_2 dz + \frac{1}{\gamma}\xi_1\xi_2\big|_{z=1}\right] = 0 \tag{E.53}$$

and since γ is assumed to be one, it becomes

$$\left(\lambda_2^2 - \lambda_1^2\right)\left[\int_0^1 \xi_1\xi_2 dz + \xi_1\xi_2\big|_{z=1}\right] = 0 \tag{E.54}$$

which is same as

$$\left(\lambda_2^2 - \lambda_1^2\right)\left\langle \xi_1, \xi_2 \right\rangle = 0 \tag{E.55}$$

where $\langle \xi_1, \xi_2 \rangle$ is the inner product of ξ_1 and ξ_2, which means

$$\langle \xi_1, \xi_2 \rangle = \int_0^1 \xi_1 \xi_2 dz + \xi_1 \xi_2 \big|_{z=1} \tag{E.56}$$

From Equation (E.55) we can see that when $\lambda_1^2 \neq \lambda_2^2$ then $\langle \xi_1, \xi_2 \rangle$ must be zero (*i.e.*, ξ_1 and ξ_2 are orthogonal).

Now, by taking the inner product of Equation (E.37) on $\sin \lambda_1 z$, we get

$$\begin{aligned} \langle z(1-z), \sin \lambda_1 z \rangle = B_1 \langle \sin \lambda_1 z, \sin \lambda_1 z \rangle \\ + B_2 \langle \sin \lambda_2 z, \sin \lambda_1 z \rangle \\ + B_3 \langle \sin \lambda_3 z, \sin \lambda_1 z \rangle + \ldots \end{aligned} \tag{E.57}$$

where, at $m \neq n$, $\langle \sin \lambda_m z, \sin \lambda_n z \rangle$ must vanish. Therefore, from Equation (E.57) B_1 can be obtained and it is

$$B_1 = \frac{\langle z(1-z), \sin \lambda_1 z \rangle}{\langle \sin \lambda_1 z, \sin \lambda_1 z \rangle} = \frac{\int_{z=0}^{z=1} z(1-z) \sin \lambda_1 z dz + z(1-z) \sin \lambda_1 z \big|_{z=1}}{\int_{z=0}^{z=1} \sin^2 \lambda_1 z dz + \sin^2 \lambda_1 z \big|_{z=1}} \tag{E.58}$$

and because the value of λ_1 is known[1], $\lambda_1 = 0.86$, then B_1 can be calculated and it is $B_1 = 0.087$, which is close to the value found by the previous method (*i.e.*, sequence of terminations method), and tabulated in Table E-2. On the same measure, B_2 can be obtained by taking the inner product of Equation (E.37) on $\sin \lambda_2 z$, this gives

$$\begin{aligned} \langle z(1-z), \sin \lambda_2 z \rangle = B_1 \langle \sin \lambda_1 z, \sin \lambda_2 z \rangle \\ + B_2 \langle \sin \lambda_2 z, \sin \lambda_2 z \rangle \\ + B_3 \langle \sin \lambda_3 z, \sin \lambda_2 z \rangle + \ldots \end{aligned} \tag{E.59}$$

---

[1] The values of λ_1 to λ_6 are illustrated in Table E-1.

then B_2 can be written as

$$B_2 = \frac{\langle z(1-z), \sin \lambda_2 z \rangle}{\langle \sin \lambda_2 z, \sin \lambda_2 z \rangle} = \frac{\int_{z=0}^{z=1} z(1-z) \sin \lambda_2 z dz + z(1-z) \sin \lambda_2 z \big|_{z=1}}{\int_{z=0}^{z=1} \sin^2 \lambda_2 z dz + \sin^2 \lambda_2 z \big|_{z=1}} \tag{E.60}$$

and because λ_2 is known, $\lambda_2 = 3.42$, then B_2 can be calculated and it is $B_2 = 0.226$. Again, this value is close to the one obtained from the previous calculation and it is presented in Table E-2. The values of $B_3, B_4, B_5, B_6, \ldots$ can be found by following the same steps and they are illustrated in Table E-3.

Table E-3. Obtained values of B_1 to B_6 via the two methods.

B_n's	Sequence of terminations method	Inner product method
B_1	0.089	0.087
B_2	0.224	0.226
B_3	-0.006	-0.007
B_4	0.011	0.011
B_5	-5×10^{-4}	-1.03×10^{-3}
B_6	0.002	0.0026

From this table we see that the first method, the sequence of terminations method, works properly and it gives very close results to those obtained via the inner product method.

APPENDIX F
DETERMINING THE PRESSURE GRADIENT

To find $\nabla P_o, \nabla P_1$ and ∇P_2 we turn to the flow equation

$$\vec{V} = -\nabla P$$

and on perturbing this equation we get: At order ε^o

$$\vec{V}_o = -\nabla P_o \tag{F.1}$$

At order ε^1

$$\vec{V}_1 = -\nabla P_1 \tag{F.2}$$

At order ε^2

$$\vec{V}_2 = -\nabla P_2 \tag{F.3}$$

Then, by taking the divergence of these equations we obtain: For order ε^o,

$$\nabla \cdot \vec{V}_o = -\nabla^2 P_o \tag{F.4}$$

For order ε^1,

$$\nabla \cdot \vec{V}_1 = -\nabla^2 P_1 \tag{F.5}$$

For order ε^2,

$$\nabla \cdot \vec{V}_2 = -\nabla^2 P_2 \tag{F.6}$$

but from the equation of continuity we know that

$$\nabla \cdot \vec{V} = 0 \tag{F.7}$$

hence,

$$\nabla \cdot \vec{V}_o = 0,\ \nabla \cdot \vec{V}_1 = 0 \text{ and } \nabla \cdot \vec{V}_2 = 0$$

Then to solve for $\nabla P_o, \nabla P_1$ and ∇P_2 we must solve the following set of problems: First at order ε^o, which is the symmetric annular region, we have

$$\nabla^2 P_o = 0 \tag{F.8}$$

where

$$P_o(r_o = R_{out}) = P^L \tag{F.9}$$

and

$$P_o(r_o = 1) = P^H \tag{F.10}$$

For order ε^1 the equation is

$$\nabla^2 P_1 = 0 \tag{F.11}$$

where

$$P_1(r_o = R_{out}) = 0 \tag{F.12}$$

and

$$P_1(r_o = 1) = -R_1 \left.\frac{dP_o}{dr_o}\right|_{r_o=1} \tag{F.13}$$

For order ε^2, the equation becomes

$$\nabla^2 P_2 = 0 \tag{F.14}$$

where

$$P_2(r_o = R_{out}) = 0 \tag{F.15}$$

and

$$P_2(r_o = 1) = -2R_1 \left.\frac{dP_1}{dr_o}\right|_{r_o=1} - R_1^2 \left.\frac{d^2 P_o}{dr_o^2}\right|_{r_o=1} - R_2 \left.\frac{dP_o}{dr_o}\right|_{r_o=1} \tag{F.16}$$

To explain where Equations (F.13) and (F.16) come from, notice that the expansion of P at $r = R(\theta,\varepsilon)$, is given by

$$P(R,\theta) = P_o + \varepsilon\left(P_1 + R_1\frac{dP_o}{dr_o}\right) + \frac{\varepsilon^2}{2}\left(P_2 + 2R_1\frac{dP_1}{dr_o} + R_1^2\frac{d^2P_o}{dr_o^2} + R_2\frac{dP_o}{dr_o}\right) + \ldots$$

and that $P(R,\theta)$ must vanish for all values of ε.

Begin by solving the problem at order ε^o where we have

$$\nabla^2 P_o = \frac{1}{r_o}\frac{\partial}{\partial r_o}(r_o\frac{\partial P_o}{\partial r_o}) = 0 \tag{F.17}$$

The solution of this problem must satisfy the boundary conditions given by Equations (F.9) and (F.10). Solving this problem the pressure in the annular region is obtained as

$$P_o = P^H - \frac{\left(P^H - P^L\right)}{\ln\left(R_{out}\right)}\ln\left(r_o\right) \tag{F.18}$$

Moreover,

$$\frac{dP_o}{dr_o} = -\frac{\left(P^H - P^L\right)}{\ln\left(R_{out}\right)}\frac{1}{r_o} \tag{F.19}$$

and

$$\frac{d^2P_o}{dr_o^2} = \frac{\left(P^H - P^L\right)}{\ln\left(R_{out}\right)}\frac{1}{r_o^2} \tag{F.20}$$

These derivatives will be useful later on.

For order ε^1, the problem is given as

$$\nabla^2 P_1 = \left[\frac{1}{r_o}\frac{\partial}{\partial r_o}(r_o\frac{\partial}{\partial r_o}) + \frac{1}{r_o^2}\frac{\partial^2}{\partial\theta_o^2}\right]P_1 = 0 \tag{F.21}$$

and its solution must satisfy the boundary conditions given by Equations (F.12) and (F.13). Then, because of R_1, the solution to the P_1 problem (*i.e.*, Equation (F.21)), can be written as

$$P_1 = \left(A_1 r_o + \frac{B_1}{r_o} \right) \cos\theta_o \tag{F.22}$$

where A_1 and B_1 must satisfy

$$A_1 R_{out} + \frac{B_1}{R_{out}} = 0 \tag{F.23}$$

and

$$A_1 + B_1 = -\left.\frac{dP_o}{dr_o}\right|_{r_o=1} \tag{F.24}$$

By solving Equations (F.23) and (F.24) for A_1 and B_1, P_1 found to be

$$P_1 = \cos\theta_o \frac{1}{\left(R_{out}^2 - 1\right)} \left(r_o - \frac{R_{out}^2}{r_o} \right) \left.\frac{dP_o}{dr_o}\right|_{r_o=1} \tag{F.25}$$

and by this, there obtains

$$\frac{dP_1}{dr_o} = \cos\theta_o \frac{1}{\left(R_{out}^2 - 1\right)} \left(1 + \frac{R_{out}^2}{r_o^2} \right) \left.\frac{dP_o}{dr_o}\right|_{r_o=1} \tag{F.26}$$

and

$$\frac{dP_1}{d\theta_o} = -\sin\theta_o \frac{1}{\left(R_{out}^2 - 1\right)} \left(r_o - \frac{R_{out}^2}{r_o} \right) \left.\frac{dP_o}{dr_o}\right|_{r_o=1} \tag{F.27}$$

which are needed in the P_2 problem.

To solve the P_2 problem (*i.e.*, Equation (F.14)), first substitute Equations (F.19), (F.20) and (F.26) into Equation (F.16) then expand the right-hand side of Equation (F.16) in terms of the functions[1] 1, $\cos\theta_o, \cos 2\theta_o$. Doing this Equation (F.16) reduces to

$$P_2\left(r_o=1\right)=\frac{-1}{\left(R_{out}^2-1\right)}\left[2+\left(1+R_{out}^2\right)\cos 2\theta_o\right]\left.\frac{dP_o}{dr_o}\right|_{r_o=1} \tag{F.28}$$

Then, from Equations (F.15) and (F.28), the solution to the P_2 problem can be written as

$$P_2=A_o+B_o\ln\left(r_o\right)+\left(A_2r_o^2+\frac{B_2}{r_o^2}\right)\cos 2\theta_o \tag{F.29}$$

where A_o and B_o must satisfy

$$A_o+B_o\ln\left(R_{out}\right)=0 \tag{F.30}$$

and

$$A_o=\frac{-2}{\left(R_{out}^2-1\right)}\left.\frac{dP_o}{dr_o}\right|_{r_o=1} \tag{F.31}$$

and where A_2 and B_2 must satisfy

$$A_2R_{out}^2+\frac{B_2}{R_{out}^2}=0 \tag{F.32}$$

and

$$A_2+B_2=\frac{-\left(1+R_{out}^2\right)}{\left(R_{out}^2-1\right)}\left.\frac{dP_o}{dr_o}\right|_{r_o=1} \tag{F.33}$$

---

[1] Substitute for $\cos^2\theta_o=\frac{1}{2}+\frac{1}{2}\cos 2\theta_o$ and for $\sin^2\theta_o=\frac{1}{2}-\frac{1}{2}\cos 2\theta_o$.

By solving Equations (F.30) and (F.31) for A_o and B_o, and Equations (F.32) and (F.33) for A_2 and B_2, P_2 is found to be

$$P_2 = -\left[\frac{2}{\left(R_{out}^2 - 1\right)\ln\left(R_{out}\right)}\ln\left(\frac{R_{out}}{r_o}\right) + \frac{1}{\left(R_{out}^2 - 1\right)^2}\left(\frac{R_{out}^4}{r_o^2} - r_o^2\right)\cos 2\theta_o\right]\left.\frac{dP_o}{dr_o}\right|_{r_o=1} \qquad \text{(F.34)}$$

and by this, there obtains

$$\frac{dP_2}{dr_o} = \left[\frac{2}{\left(R_{out}^2 - 1\right)\ln\left(R_{out}\right)r_o} + \frac{2}{\left(R_{out}^2 - 1\right)^2}\left(\frac{R_{out}^4}{r_o^3} + r_o\right)\cos 2\theta_o\right]\left.\frac{dP_o}{dr_o}\right|_{r_o=1} \qquad \text{(F.35)}$$

and

$$\frac{dP_2}{d\theta_o} = \left[\frac{2}{\left(R_{out}^2 - 1\right)^2}\left(\frac{R_{out}^4}{r_o^2} - r_o^2\right)\sin 2\theta_o\right]\left.\frac{dP_o}{dr_o}\right|_{r_o=1} \qquad \text{(F.36)}$$

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